FUNCTIONS

6. POLYNOMIALS AND INVERSE FUNCTIONS

You were introduced to functions and polynomials in Chapter 4, *Functions*. In this Mathematics Extension 1 chapter, you will study polynomials in more detail and look at inverse functions.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- EXT1 divide polynomials and write them as products of their factors
- EXT1 understand and apply the remainder and factor theorems
- EXT1 solve polynomial equations
- EXT1 draw polynomial graphs using intercepts and limiting behaviour
- EXT1 understand multiplicity of roots and their effect on graphs
- EXT1 find and graph inverses of functions and identify whether the inverse is also a function
- EXT1 understand how to restrict the domain of a function so that its inverse is a function
- EXT1 understand properties of inverse functions

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EXT1 TERMINOLOGY

dividend: In division, the dividend is the polynomial or number being divided

divisor: In division, the divisor is the number or polynomial that divides another of the same type

factor theorem: The theorem that states that a polynomial $P(x)$ has a factor $x - k$ if and only if $P(k) = 0$

horizontal line test: A test that determines whether the inverse of a function is a function: any horizontal line drawn on the graph of the original function should cut the graph at most once

inverse function: An inverse function undoes the original function and can be shown by exchanging the x and y values of the original function

monotonic decreasing: Always decreasing

monotonic increasing: Always increasing

multiplicity: If $P(x) = (x - k)^r Q(x)$ where $Q(x) \neq 0$ and r is a positive integer, then the root $x = k$ has multiplicity r

quotient: The result when dividing two numbers or polynomials

remainder: A number or polynomial that is left over after dividing two numbers or polynomials

remainder theorem: The theorem that states that if a polynomial $P(x)$ is divided by $x - k$, then the remainder is given by $P(k)$

restricted domain: Domain restricted to the x values that will make the inverse relation a function

EXT1 6.01 Division of polynomials

Long division is a way to divide by a two-digit number without using a calculator. We can also use this method to divide polynomials. This allows us to factorise polynomials.

INVESTIGATION

LONG DIVISION

Study this example of long division: 5715 ÷ 48.

$$\begin{array}{r}
119\text{ r}3 \\
48\overline{)5715} \\
\underline{48} \\
91 \\
\underline{48} \\
435 \\
\underline{432} \\
3
\end{array}$$

$$\frac{5715}{48} = 119 + \frac{3}{48}$$

This means $\frac{5715}{48} \times 48 = 119 \times 48 + \frac{3}{48} \times 48$.

So 5715 = 48 × 119 + 3. (Check this on your calculator.)

The number 5715 is called the dividend, the 48 is the divisor, 119 is the quotient and 3 is the remainder.

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In Chapter 4, *Functions*, we learned that a polynomial is an expression in the form $P(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$, where n is a positive integer or zero. If we divide a polynomial $P(x)$ by $A(x)$, we can write $P(x)$ in the form $\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)}$ where $Q(x)$ is the quotient and $R(x)$ is the remainder.

$$\frac{P(x)}{A(x)} \times A(x) = Q(x) \times A(x) + \frac{R(x)}{A(x)} \times A(x)$$

$$P(x) = A(x)Q(x) + R(x)$$

Dividing polynomials

A polynomial $P(x)$ can be written as $P(x) = A(x)Q(x) + R(x)$

where $P(x)$ is the **dividend**, $A(x)$ is the **divisor**, $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

The degree of the remainder $R(x)$ is always less than the degree of the divisor $A(x)$.

EXAMPLE 1

a **i** Divide $P(x) = 3x^4 - x^3 + 7x^2 - 2x + 3$ by $x - 2$.

ii Hence write $P(x)$ in the form $P(x) = A(x)Q(x) + R(x)$.

iii Show that $P(2)$ is equal to the remainder.

b For each pair of polynomials, divide $P(x)$ by $A(x)$ and then write $P(x)$ in the form $P(x) = A(x)Q(x) + R(x)$.

i $P(x) = x^3 - 3x^2 + x + 4$, $A(x) = x^2 - x$

ii $P(x) = x^5 + x^3 + 5x^2 - 6x + 15$, $A(x) = x^2 + 3$

Solution

a **i** Step 1: Dividing $3x^4$ by x gives $3x^3$.

$$\begin{array}{rl} & 3x^3 \\ x-2 \hspace{-4pt} & \overline{)\,3x^4 - x^3 + 7x^2 - 2x + 3} \end{array}$$

Step 2: Multiply $3x^3$ by $(x - 2)$ and find the remainder by subtraction.

$3x^3(x - 2) = 3x^4 - 6x^3$

$$\begin{array}{rl} & 3x^3 \\ x-2 \hspace{-4pt} & \overline{)\,3x^4 - x^3 + 7x^2 - 2x + 3} \\ & \underline{3x^4 - 6x^3} \\ & \phantom{3x^4 - {}} 5x^3 \end{array}$$

Step 3: Bring down the $7x^2$ and next divide $5x^3$ by x to give $5x^2$.

$$\begin{array}{rl} & 3x^3 + 5x^2 \\ x-2 \hspace{-4pt} & \overline{)\,3x^4 - x^3 + 7x^2 - 2x + 3} \\ & \underline{3x^4 - 6x^3} \\ & \phantom{3x^4 - {}} 5x^3 + 7x^2 \end{array}$$

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Step 4: Multiply $5x^2$ by $(x - 2)$ and find the remainder by subtraction.

$5x^2(x - 2) = 5x^3 - 10x^2$

$$\begin{array}{rl}
 & 3x^3 + 5x^2 \\
x-2 & \overline{)\,3x^4 - x^3 + 7x^2 - 2x + 3} \\
 & \underline{3x^4 - 6x^3} \\
 & \qquad 5x^3 + 7x^2 \\
 & \qquad \underline{5x^3 - 10x^2} \\
 & \qquad\qquad 17x^2
\end{array}$$

Continue this way until we have a number (67) as the remainder.

$$\begin{array}{rl}
 & 3x^3 + 5x^2 + 17x + 32 \\
x-2 & \overline{)\,3x^4 - x^3 + 7x^2 - 2x + 3} \\
 & \underline{3x^4 - 6x^3} \\
 & \qquad 5x^3 + 7x^2 \\
 & \qquad \underline{5x^3 - 10x^2} \\
 & \qquad\qquad 17x^2 - 2x \\
 & \qquad\qquad \underline{17x^2 - 34x} \\
 & \qquad\qquad\qquad 32x + 3 \\
 & \qquad\qquad\qquad \underline{32x - 64} \\
 & \qquad\qquad\qquad\qquad 67
\end{array}$$

ii $P(x) = 3x^4 - x^3 + 7x^2 - 2x + 3$ is the dividend.

$A(x) = x - 2$ is the divisor.

$Q(x) = 3x^3 + 5x^2 + 17x + 32$ is the quotient.

$R(x) = 67$ is the remainder.

$P(x) = A(x)Q(x) + R(x)$.

So $3x^4 - x^3 + 7x^2 - 2x + 3 = (x - 2)(3x^3 + 5x^2 + 17x + 32) + 67$.

iii $P(2) = 3(2)^4 - (2)^3 + 7(2)^2 - 2(2) + 3$

$= 48 - 8 + 28 - 4 + 3$

$= 67$

$\therefore$ $P(2)$ is equal to the remainder.

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b **i**

$$\begin{array}{r}
x-2 \\
x^2-x\overline{\smash{)}x^3-3x^2+x+4} \\
\underline{x^3-x^2} \\
-2x^2+x \\
\underline{-2x^2+2x} \\
-x+4
\end{array}$$

$(x^3 - 3x^2 + x + 4) \div (x^2 - x) = x - 2$, remainder $-x + 4$

So $x^3 - 3x^2 + x + 4 = (x - 2)(x^2 - x) + (-x + 4)$

ii

$$\begin{array}{r}
x^3-2x+5 \\
x^2+3\overline{\smash{)}x^5+x^3+5x^2-6x+15} \\
\underline{x^5+3x^3} \\
-2x^3+5x^2-6x \\
\underline{-2x^3-6x} \\
5x^2+15 \\
\underline{5x^2+15} \\
0
\end{array}$$

So $x^5 + x^3 + 5x^2 - 6x + 15 = (x^3 - 2x + 5)(x^2 + 3)$

EXT1 Exercise 6.01 Division of polynomials

Divide each pair of polynomials and write the dividend in the form $P(x) = A(x)Q(x) + R(x)$.

1. $(3x^2 + 2x + 5) \div (x + 4)$
2. $(x^2 + 5x - 2) \div (x + 1)$
3. $(x^2 - 7x + 4) \div (x - 1)$
4. $(x^3 + x^2 + 2x - 1) \div (x - 3)$
5. $(4x^2 + 2x - 3) \div (2x + 3)$
6. $(x^3 + x^2 - x - 3) \div (x - 2)$
7. $(x^4 - x^3 - 2x^2 + x - 3) \div (x + 4)$
8. $(4x^3 - 2x^2 + 6x - 1) \div (2x + 1)$
9. $(3x^5 - 2x^4 - 3x^3 + x^2 - x - 1) \div (x + 2)$
10. $(x^4 - 2x^2 + 5x + 4) \div (x - 3)$
11. $(2x^3 + 4x^2 - x + 8) \div (x^2 + 3x + 2)$
12. $(x^4 - 2x^3 + 4x^2 + 2x + 5) \div (x^2 + 2x - 1)$
13. $(3x^5 - 2x^3 + x - 1) \div (x + 1)$
14. $(x^3 - 3x^2 + 3x - 1) \div (x^2 + 5)$
15. $(2x^4 - 5x^3 + 2x^2 + 2x - 5) \div (x^2 - 2x)$

WS
The remainder theorem

WS
Factorising polynomials

EXT1 6.02 Remainder and factor theorems

Remainder theorem

If a polynomial $P(x)$ is divided by $x - k$, then the remainder is $P(k)$.

Proof

$P(x) = A(x)Q(x) + R(x)$ where $A(x) = x - k$

$P(x) = (x - k)Q(x) + R(x)$

The degree of $A(x)$ is 1, so the degree of $R(x)$ must be 0.

So $R(x) = c$ where c is a constant.

$\therefore\ P(x) = (x - k)Q(x) + c$

Substituting $x = k$:

$$\begin{aligned} P(k) &= (k - k)Q(k) + c \\ &= 0 \cdot Q(k) + c \\ &= c \end{aligned}$$

So $P(k)$ is the remainder.

EXAMPLE 2

a Find the remainder when $3x^4 - 2x^2 + 5x + 1$ is divided by $x - 2$.

b Evaluate m if the remainder is 4 when $2x^4 + mx + 5$ is divided by $x + 3$.

Solution

a When $P(x)$ is divided by $x - 2$ the remainder is $P(2)$.

$$\begin{aligned} P(2) &= 3(2)^4 - 2(2)^2 + 5(2) + 1 \\ &= 51 \end{aligned}$$

So the remainder is 51.

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b The remainder when $P(x)$ is divided by $x + 3$ is $P(-3)$ since $x + 3 = x - (-3)$.

So $\quad P(-3) = 4$

$$
\begin{aligned}
2(-3)^4 + m(-3) + 5 &= 4 \\
162 - 3m + 5 &= 4 \\
167 - 3m &= 4 \\
167 &= 3m + 4 \\
163 &= 3m \\
54\tfrac{1}{3} &= m
\end{aligned}
$$

The **factor theorem** is a direct result of the **remainder theorem**.

Factor theorem

For a polynomial $P(x)$, if $P(k) = 0$ then $x - k$ is a factor of the polynomial.

Proof

$P(x) = (x - k)\,Q(x) + R(x)$.

The remainder theorem states that when $P(x)$ is divided by $x - k$, the remainder is $P(k)$.

So $P(x) = (x - k)\,Q(x) + P(k)$.

But if $P(k) = 0$:

$$
\begin{aligned}
P(x) &= (x - k)\,Q(x) + 0 \\
&= (x - k)\,Q(x)
\end{aligned}
$$

So $x - k$ is a factor of $P(x)$.

The converse is also true:

Converse of the factor theorem

For a polynomial $P(x)$, if $x - k$ is a factor of the polynomial, then $P(k) = 0$.

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EXAMPLE 3

a Show that $x - 1$ is a factor of $P(x) = x^3 - 7x^2 + 8x - 2$.

b Divide $P(x)$ by $x - 1$ and write $P(x)$ in the form $P(x) = (x - 1)Q(x)$.

Solution

a The remainder when dividing the polynomial by $x - 1$ is $P(1)$.

$$P(1) = 1^3 - 7(1)^2 + 8(1) - 2$$
$$= 0$$

So $x - 1$ is a factor of $P(x)$.

b

$$\begin{array}{r|l} & x^2 - 6x + 2 \\ \hline x - 1 & x^3 - 7x^2 + 8x - 2 \\ & \underline{x^3 - x^2} \\ & -6x^2 + 8x \\ & \underline{-6x^2 + 6x} \\ & 2x - 2 \\ & \underline{2x - 2} \\ & 0 \end{array}$$

So $x^3 - 7x^2 + 8x - 2 = (x - 1)(x^2 - 6x + 2)$.

Some properties of polynomials come from the remainder and factor theorems.

The **zeros** of the polynomial $P(x)$ are those values of x for which $P(x) = 0$.

Properties of polynomials

- If polynomial $P(x)$ has n distinct zeros $k_1, k_2, k_3, \ldots, k_n$, then $(x - k_1)(x - k_2)(x - k_3) \ldots (x - k_n)$ is a factor of $P(x)$.
- If polynomial $P(x)$ has degree n and n distinct zeros $k_1, k_2, k_3, \ldots, k_n$, then $P(x) = a_n(x - k_1)(x - k_2)(x - k_3) \ldots (x - k_n)$.
- A polynomial of degree n cannot have more than n distinct real zeros.
- A polynomial of degree n with more than n distinct real zeros is the **zero polynomial** $P(x) = 0x^n + 0x^{n-1} + \ldots + 0x^2 + 0x + 0$.
- If 2 polynomials of degree n are equal for more than n distinct values of x, then the coefficients of like powers of x are equal:

  if $a_nx^n + \ldots + a_2x^2 + a_1x + a_0 \equiv b_nx^n + \ldots + b_2x^2 + b_1x + b_0$, then $a_n = b_n, \ldots, a_2 = b_2, a_1 = b_1, a_0 = b_0$.

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EXAMPLE 4

If a polynomial has degree 2, show that it cannot have 3 zeros.

Solution

Let $P(x) = a_2x^2 + a_1x + a_0$ where $a_2 \neq 0$.

Assume $P(x)$ has 3 zeros, k_1, k_2 and k_3.

Then $(x - k_1)(x - k_2)(x - k_3)$ is a factor of the polynomial.

$\therefore P(x) = (x - k_1)(x - k_2)(x - k_3)Q(x)$

But this polynomial has degree 3 and $P(x)$ only has degree 2.

So $P(x)$ cannot have 3 zeros.

EXAMPLE 5

Write $x^3 - 2x^2 + 5$ in the form $ax^3 + b(x + 3)^2 + c(x + 3) + d$.

Solution

$$\begin{aligned}
ax^3 + b(x + 3)^2 + c(x + 3) + d &= ax^3 + b(x^2 + 6x + 9) + c(x + 3) + d \\
&= ax^3 + bx^2 + 6bx + 9b + cx + 3c + d \\
&= ax^3 + bx^2 + (6b + c)x + 9b + 3c + d
\end{aligned}$$

For $x^3 - 2x^2 + 5 \equiv ax^3 + bx^2 + (6b + c)x + 9b + 3c + d$:

by equating coefficients

$$\begin{aligned}
a &= 1 && [1] \\
b &= -2 && [2] \\
6b + c &= 0 && [3] \\
9b + 3c + d &= 5 && [4]
\end{aligned}$$

Substitute [2] into [3]:

$$\begin{aligned}
6(-2) + c &= 0 \\
-12 + c &= 0 \\
c &= 12
\end{aligned}$$

Substitute $b = -2$ and $c = 12$ into [4]:

$$\begin{aligned}
9(-2) + 3(12) + d &= 5 \\
-18 + 36 + d &= 5 \\
d &= -13
\end{aligned}$$

$\therefore x^3 - 2x^2 + 5 \equiv x^3 - 2(x + 3)^2 + 12(x + 3) - 13$.

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Factorising polynomials

Factorising polynomials

If $x - k$ is a factor of polynomial $P(x)$, then k is a factor of the constant term of the polynomial.

You already use this property to factorise quadratic trinomials of the form $ax^2 + bx + c$.

Proof

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ where $a_n \neq 0$.

If $x - k$ is a factor of $P(x)$, then:

$P(x) = (x - k)Q(x)$ where $Q(x)$ has degree $n - 1$.

$$\begin{aligned}
P(x) &= (x - k)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_2x^2 + b_1x + b_0) \text{ where } b_{n-1} \neq 0 \\
&= xb_{n-1}x^{n-1} + xb_{n-2}x^{n-2} + \ldots + xb_1x + xb_0 - kb_{n-1}x^{n-1} - kb_{n-2}x^{n-2} - \ldots \\
&\quad - kb_2x^2 - kb_1x - kb_0 \\
&= b_{n-1}x^n + b_{n-2}x^{n-1} + \ldots + b_1x^2 + b_0x - kb_{n-1}x^{n-1} - kb_{n-2}x^{n-2} - \ldots - kb_2x^2 - kb_1x - kb_0 \\
&= b_{n-1}x^n + (b_{n-2} - kb_{n-1})x^{n-1} + \ldots + (b_1 - k)x^2 + (b_0 - k)x - kb_0
\end{aligned}$$

$\therefore a_0 = -kb_0$

So k is a factor of a_0.

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EXAMPLE 6

Factorise each polynomial.

a $P(x) = x^3 + 3x^2 - 4x - 12$

b $P(x) = x^3 + 3x^2 + 5x + 15$

Solution

a Try factors of the constant term, -12 (that is, $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$).

Substitute these into $P(x)$ until you find one where $P(k) = 0$.

$P(1) = 1^3 + 3(1)^2 - 4(1) - 12 = -12 \neq 0$

$\therefore$ $x - 1$ is not a factor of $P(x)$.

$P(2) = 2^3 + 3(2)^2 - 4(2) - 12 = 0$

$\therefore$ $x - 2$ is a factor of $P(x)$.

Divide $P(x)$ by $x - 2$ to find other factors:

$$\begin{array}{r}
x^2 + 5x + 6 \\
x-2\overline{)x^3 + 3x^2 - 4x - 12} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
5x^2 - 4x \qquad \\
\underline{5x^2 - 10x} \qquad \\
6x - 12 \\
\underline{6x - 12} \\
0
\end{array}$$

$\therefore P(x) = (x - 2)(x^2 + 5x + 6)$

$= (x - 2)(x + 2)(x + 3)$

b Try factors of 15 (that is, $\pm1, \pm3, \pm5, \pm15$).

$P(-3) = (-3)^3 + 3(-3)^2 + 5(-3) + 15 = 0$

$\therefore$ $x + 3$ is a factor of $f(x)$.

Divide $P(x)$ by $x + 3$ to find other factors:

$$\begin{array}{r}
x^2 \qquad + 5 \\
x+3\overline{)x^3 + 3x^2 + 5x + 15} \\
\underline{x^3 + 3x^2} \qquad\qquad \\
0 + 5x + 15 \\
\underline{5x + 15} \\
0
\end{array}$$

$\therefore P(x) = (x + 3)(x^2 + 5)$

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EXT1 Exercise 6.02 Remainder and factor theorems

1 Use the remainder theorem to find the remainder in each division.

a $(x^3 - 2x^2 + x + 5) \div (x - 4)$
b $(x^2 + 5x + 3) \div (x + 2)$
c $(2x^3 - 4x - 1) \div (x + 3)$
d $(3x^5 + 2x^2 - x + 4) \div (x - 5)$
e $(5x^3 + 2x^2 + 2x - 9) \div (x - 1)$
f $(x^4 - x^3 + 3x^2 - x - 1) \div (x + 2)$
g $(2x^2 + 7x - 2) \div (x + 7)$
h $(x^7 + 5x^3 - 1) \div (x - 3)$
i $(2x^6 - 3x^2 + x + 4) \div (x + 5)$
j $(3x^4 - x^3 - x^2 - x - 7) \div (x + 1)$

2 Find the value of k if:

a the remainder is 3 when $5x^2 - 10x + k$ is divided by $x - 1$
b the remainder is -14 when $x^3 - (k - 1)x^2 + 5kx + 4$ is divided by $x + 2$
c the remainder is 0 when $2x^5 + 7x^2 + 1 + k$ is divided by $x + 6$
d $2x^4 - kx^3 + 3x^2 + x - 3$ is divisible by $x - 3$
e the remainder is 25 when $2x^4 - 3x^2 + 5$ is divided by $x - k$.

3 **a** Find the remainder when $f(x) = x^3 - 4x^2 + x + 6$ is divided by $x - 2$.
b Is $x - 2$ a factor of $f(x)$?
c Divide $x^3 - 4x^2 + x + 6$ by $x - 2$.
d Factorise $f(x)$ fully and write $f(x)$ as a product of its factors.

4 **a** Show that $x + 3$ is a factor of $P(x) = x^4 + 3x^3 - 9x^2 - 27x$.
b Divide $P(x)$ by $x + 3$ and write $P(x)$ as a product of its factors.

5 The remainder is 89 when $P(x) = ax^3 - 4bx^2 + x - 4$ is divided by $x - 3$, and the remainder is -3 when $P(x)$ is divided by $x + 1$. Find the values of a and b.

6 When $f(x) = ax^2 - 3x + 1$ and $g(x) = x^3 - 3x^2 + 2$ are divided by $x + 1$ they leave the same remainder. Find the value of a.

7 **a** Show that $x - 3$ is not a factor of $P(x) = x^5 - 2x^4 + 7x^2 - 3x + 5$.
b Find a value of k such that $x - 3$ is a factor of $Q(x) = 2x^3 - 5x + k$.

8 The polynomial $P(x) = x^3 + ax^2 + bx + 2$ has factors $x + 1$ and $x - 2$.

a Find the values of a and b.
b Write $P(x)$ as a product of its factors.

9 **a** The remainder when $f(x) = ax^4 + bx^3 + 15x^2 + 9x + 2$ is divided by $x - 2$ is 216, and $x + 1$ is a factor of $f(x)$. Find a and b.
b Divide $f(x)$ by $x + 1$ and write the polynomial in the form $f(x) = (x + 1)g(x)$.
c Show that $x + 1$ is a factor of $g(x)$.
d Write $f(x)$ as a product of its factors.

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10 Write each polynomial as a product of its factors.

a $P(x) = x^2 - 2x - 8$

b $P(x) = x^3 + x^2 - 2x$

c $f(x) = x^3 + x^2 - 10x + 8$

d $g(x) = x^3 + 4x^2 - 11x - 30$

e $G(x) = x^3 - 11x^2 + 31x - 21$

f $P(x) = x^3 - 12x^2 + 17x + 90$

g $Q(x) = x^3 - 7x^2 + 16x - 12$

h $R(x) = x^4 + 6x^3 + 9x^2 + 4x$

11 **a** Write $P(x) = x^3 - 7x + 6$ as a product of its factors.

b What are the zeros of $P(x)$?

c Is $(x - 2)(x + 3)$ a factor of $P(x)$?

12 If $f(x) = x^4 + 10x^3 + 23x^2 - 34x - 120$ has zeros -5 and 2:

a show that $(x + 5)(x - 2)$ is a factor of $f(x)$

b write $f(x)$ as a product of its linear factors.

13 If $P(x) = x^4 + 3x^3 - 13x^2 - 51x - 36$ has zeros -3 and 4, write $P(x)$ as a product of its linear factors.

14 **a** Show that $P(x) = x^3 - 3x^2 - 34x + 120$ has zeros -6 and 5.

b Write $P(x)$ as a product of its linear factors.

15 Evaluate a, b, c and d if:

a $x^2 + 4x - 3 \equiv a(x + 1)^2 + b(x + 1) + c$

b $2x^2 - 3x + 1 \equiv a(x + 2)^2 + b(x + 2) + c$

c $x^2 - x - 2 \equiv a(x - 1)^2 + b(x - 1) + c$

d $x^2 + x + 6 \equiv a(x - 3)^2 + b(x - 3) + c$

e $3x^2 - 5x - 2 \equiv a(x + 1)^2 + b(x - 1) + c$

f $x^3 + 3x^2 - 2x + 1 \equiv ax^3 + b(x - 1)^2 + cx + d$

The congruency symbol ≡ means 'is identical to' when applied to algebra.

16 A monic polynomial of degree 3 has zeros -3, 0 and 4. Find the polynomial.

17 Polynomial $P(x) = ax^3 - bx^2 + cx - 8$ has zeros 2 and -1, and $P(3) = 28$. Evaluate a, b and c.

18 A polynomial with leading term $2x^4$ has zeros -2, 0, 1 and 3. Find the polynomial.

19 Show that a polynomial of degree 2 cannot have 3 zeros.

20 Show that a polynomial of degree 3 cannot have 4 zeros.

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EXT1 6.03 Polynomial equations

$P(x)$ is a **polynomial** while $P(x) = 0$ is a **polynomial equation**.

The solutions to $P(x) = 0$ are called the **roots** of the equation or the **zeros** of the polynomial $P(x)$.

EXAMPLE 7

a Find all zeros of $P(x) = x^3 - 7x + 6$.

b Find the roots of $x^4 + 4x^3 - 7x^2 - 10x = 0$.

Solution

a Factorise $P(x)$ by trying factors of the constant term, 6 (that is, $\pm 1, \pm 2, \pm 3, \pm 6$).

$P(1) = 1^3 - 7(1) + 6 = 0$

So $x - 1$ is a factor of $P(x)$.

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x-1 & x^3 \qquad - 7x + 6 \\
 & \underline{x^3 - x^2} \\
 & x^2 - 7x \\
 & \underline{x^2 - x + 6} \\
 & -6x + 6 \\
 & \underline{-6x + 6} \\
 & 0
\end{array}
$$

$P(x) = (x - 1)(x^2 + x - 6)$

$= (x - 1)(x + 3)(x - 2)$

For zeros, $P(x) = 0$:

$(x - 1)(x + 3)(x - 2) = 0$

$x = 1, -3, 2$

b Factorising: $x^4 + 4x^3 - 7x^2 - 10x = x(x^3 + 4x^2 - 7x - 10)$

To factorise $x^3 + 4x^2 - 7x - 10$, try factors of -10:

$P(1) = 1^3 + 4(1)^2 - 7(1) - 10 = -12 \neq 0$

$P(2) = 2^3 + 4(2)^2 - 7(2) - 10 = 0$

So $x - 2$ is a factor of $P(x)$.

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$$
\begin{array}{r}
x^2 + 6x + 5 \\
x-2 \enclose{longdiv}{x^3 + 4x^2 - 7x - 10} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
6x^2 - 7x \qquad \\
\underline{6x^2 - 12x} \qquad \\
5x - 10 \\
\underline{5x - 10} \\
0
\end{array}
$$

So $x^4 + 4x^3 - 7x^2 - 10x = x(x-2)(x^2 + 6x + 5)$

$= x(x-2)(x+5)(x+1)$

Solving $x^4 + 4x^3 - 7x^2 - 10x = 0$

Roots are $x = 0, 2, -5, -1$.

EXT1 Exercise 6.03 Polynomial equations

1 Find all the zeros of each polynomial.

- **a** $P(x) = x^3 - 4x^2 + x + 6$
- **b** $R(x) = x^3 - 3x^2 - x + 3$
- **c** $P(x) = x^3 - 3x^2 - 6x + 8$
- **d** $f(x) = x^3 + x^2 - 16x + 20$
- **e** $P(x) = x^3 - 11x^2 + 23x + 35$
- **f** $P(x) = x^3 + 7x^2 - 17x + 9$
- **g** $f(x) = x^4 - 7x^2 - 6x$
- **h** $Q(x) = x^4 - x^3 - 7x^2 + x + 6$
- **i** $f(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$
- **j** $P(x) = x^4 + 3x^3 - 15x^2 - 19x + 30$

2 Find the roots of each polynomial equation.

- **a** $x^3 + x^2 - 5x + 3 = 0$
- **b** $x^3 - 3x^2 - x + 3 = 0$
- **c** $x^3 - 9x^2 + 26x - 24 = 0$
- **d** $x^3 - 2x^2 - 13x - 10 = 0$
- **e** $x^3 - 10x^2 + 23x - 14 = 0$
- **f** $x^3 - 13x - 12 = 0$
- **g** $x^4 - 9x^3 + 11x^2 + 21x = 0$
- **h** $x^4 + x^3 - 16x^2 - 4x + 48 = 0$
- **i** $x^4 - 5x^2 + 4 = 0$
- **j** $x^4 - x^3 - 13x^2 + x + 12 = 0$

3 Solve:

- **a** $2x^3 - 3x^2 - 3x + 2 = 0$
- **b** $2x^3 - 3x^2 - 2x + 3 = 0$
- **c** $5x^3 - 4x^2 - 11x - 2 = 0$
- **d** $4x^3 - 25x^2 + 49x - 30 = 0$
- **e** $6x^3 - 13x^2 + 9x - 2 = 0$

4 Find the zeros of $P(x) = x^4 - 6x^3 - 19x^2 + 84x + 180$.

5 Find the roots of $2x^4 - 5x^3 + 5x - 2 = 0$.

WS

Roots and coefficients

EXT1 6.04 Roots and coefficients of polynomial equations

Quadratic equations

If a quadratic equation $ax^2 + bx + c = 0$ has roots α and β, then the equation can be written as:

$$(x - \alpha)(x - \beta) = 0$$
$$x^2 - \beta x - \alpha x + \alpha\beta = 0$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

But $ax^2 + bx + c = 0$ can be written in monic form as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

$$\therefore\ x^2 - (\alpha + \beta)x + \alpha\beta \equiv x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\therefore\ -(\alpha + \beta) = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

giving us formulas for the sum and product of the roots in terms of the coefficients a, b and c:

Sum and product of the roots of a quadratic equation

For the quadratic equation $ax^2 + bx + c = 0$:

Sum of roots:

$$\alpha + \beta = -\frac{b}{a}$$

Product of roots:

$$\alpha\beta = \frac{c}{a}$$

EXAMPLE 8

a Find the quadratic equation that has roots $3 + \sqrt{2}$ and $3 - \sqrt{2}$.

b If α and β are the roots of $2x^2 - 6x + 1 = 0$, find:

- **i** $\alpha + \beta$
- **ii** $\alpha\beta$
- **iii** $\alpha^2 + \beta^2$

c Find the value of k if one root of $kx^2 - 7x + k + 1 = 0$ is -2.

d Evaluate p if one root of $x^2 + 2x - 5p = 0$ is double the other root.

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Solution

a $\alpha + \beta = 3 + \sqrt{2} + 3 - \sqrt{2}$

$= 6$

$\alpha\beta = (3 + \sqrt{2}) \times (3 - \sqrt{2})$

$= 3^2 - \left(\sqrt{2}\right)^2$

$= 9 - 2$

$= 7$

Substituting into $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ gives $x^2 - 6x + 7 = 0$.

b **i** $\alpha + \beta = -\dfrac{b}{a}$

$= -\dfrac{(-6)}{2}$

$= 3$

ii $\alpha\beta = \dfrac{c}{a}$

$= \dfrac{1}{2}$

iii Use $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

So $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (3)^2 - 2\left(\dfrac{1}{2}\right)$ from **i** and **ii**

$= 9 - 1$

$= 8$

c If -2 is a root of the equation then $x = -2$ satisfies the equation.

$$k(-2)^2 - 7(-2) + k + 1 = 0$$

$$4k + 14 + k + 1 = 0$$

$$5k + 15 = 0$$

$$5k = -15$$

$$k = -3$$

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d If one root is α then the other root is 2α.

Sum of roots:

$$\alpha + \beta = -\frac{b}{a}$$
$$\alpha + 2\alpha = -\frac{2}{1}$$
$$3\alpha = -2$$
$$\alpha = -\frac{2}{3}$$

Product of roots:

$$\alpha\beta = \frac{c}{a}$$
$$\alpha \times 2\alpha = \frac{-5p}{1}$$
$$2\alpha^2 = -5p$$

Substituting $\alpha = -\frac{2}{3}$:

$$2\left(-\frac{2}{3}\right)^2 = -5p$$
$$2\left(\frac{4}{9}\right) = -5p$$
$$\frac{8}{9} = -5p$$
$$p = -\frac{8}{45}$$

Cubic equations

If a cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ then:

$$(x - \alpha)(x - \beta)(x - \gamma) = 0$$
$$(x^2 - \beta x - \alpha x + \alpha\beta)(x - \gamma) = 0$$
$$x^3 - \gamma x^2 - \beta x^2 + \beta\gamma x - \alpha x^2 + \alpha\gamma x + \alpha\beta x - \alpha\beta\gamma = 0$$
$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$$

The cubic equation $ax^3 + bx^2 + cx + d = 0$ can be written in monic form as $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$.

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma \equiv x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$$

Equating coefficients gives the formulas below.

Sum and product of the roots of a cubic equation

For the cubic equation $ax^3 + bx^2 + cx + d = 0$:

Sum of roots 1 at a time:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

Sum of roots 2 at a time:

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

Do you notice a pattern in these formulas?

Product of roots:

$$\alpha\beta\gamma = -\frac{d}{a}$$

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EXAMPLE 9

a If α, β, γ are the roots of $2x^3 - 5x^2 + x - 1 = 0$, find:

i $(\alpha + \beta + \gamma)^2$ **ii** $(\alpha + 1)(\beta + 1)(\gamma + 1)$ **iii** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

b If one root of $x^3 - x^2 + 2x - 3 = 0$ is 4, find the sum and product of the other two roots.

c Solve $12x^3 + 32x^2 + 15x - 9 = 0$ given that 2 roots are equal.

Solution

a i
$$\begin{aligned}\alpha + \beta + \gamma &= -\frac{b}{a}\\ &= -\frac{(-5)}{2}\\ &= \frac{5}{2}\end{aligned}$$

$$\begin{aligned}(\alpha + \beta + \gamma)^2 &= \left(\frac{5}{2}\right)^2\\ &= 6\tfrac{1}{4}\end{aligned}$$

ii
$$\begin{aligned}&(\alpha + 1)(\beta + 1)(\gamma + 1)\\ &= (\alpha + 1)(\beta\gamma + \beta + \gamma + 1)\\ &= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \alpha + \beta\gamma + \beta + \gamma + 1\\ &= \alpha\beta\gamma + (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) + 1\end{aligned}$$

$$\begin{aligned}\alpha\beta + \alpha\gamma + \beta\gamma &= \frac{c}{a}\\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\alpha\beta\gamma &= -\frac{d}{a}\\ &= -\frac{(-1)}{2}\\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\therefore (\alpha + 1)(\beta + 1)(\gamma + 1) &= \frac{1}{2} + \frac{1}{2} + \frac{5}{2} + 1\\ &= 4\tfrac{1}{2}\end{aligned}$$

iii
$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}\\ &= \frac{\frac{1}{2}}{\frac{1}{2}}\\ &= 1\end{aligned}$$

b Roots are α, β, γ where, say, $\gamma = 4$.

$$\begin{aligned}\alpha + \beta + \gamma &= -\frac{b}{a}\\ \therefore \alpha + \beta + 4 &= 1\\ \alpha + \beta &= -3\end{aligned}$$

$$\begin{aligned}\alpha\beta\gamma &= -\frac{d}{a}\\ \alpha\beta(4) &= 3\\ \therefore \alpha\beta &= \frac{3}{4}\end{aligned}$$

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c Let the roots be α, α and β.

$$\alpha+\beta+\gamma=-\frac{b}{a}$$

$$\alpha+\alpha+\beta=-\frac{32}{12}$$

$$\therefore 2\alpha+\beta=-\frac{8}{3} \qquad [1]$$

$$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$$

$$\alpha\alpha+\alpha\beta+\alpha\beta=\frac{15}{12}$$

$$\therefore \alpha^2+2\alpha\beta=\frac{5}{4} \qquad [2]$$

$$\alpha\beta\gamma=-\frac{d}{a}$$

$$\alpha\alpha\beta=-\frac{-9}{12}$$

$$\therefore \alpha^2\beta=\frac{3}{4} \qquad [3]$$

From [1]:

$$\beta=-\frac{8}{3}-2\alpha \qquad [4]$$

Substitute in [2]:

$$\alpha^2+2\alpha\left(-\frac{8}{3}-2\alpha\right)=\frac{5}{4}$$

$$12\alpha^2+24\alpha\left(-\frac{8}{3}-2\alpha\right)=15$$

$$12\alpha^2-64\alpha-48\alpha^2=15$$

$$36\alpha^2+64\alpha+15=0$$

$$(2\alpha+3)(18\alpha+5)=0$$

$$2\alpha=-3 \qquad 18\alpha=-5$$

$$\alpha=-1\tfrac{1}{2} \qquad \alpha=-\frac{5}{18}$$

To find β, substitute each value in [4].

$\alpha=-1\frac{1}{2}$:

$$\beta=-\frac{8}{3}-2\left(-1\tfrac{1}{2}\right)$$

$$=\frac{1}{3}$$

$\alpha=-\frac{5}{18}$:

$$\beta=-\frac{8}{3}-2\left(-\frac{5}{18}\right)$$

$$=-2\tfrac{1}{9}$$

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Only one of these values for α can be correct. Test by substituting each in the LHS of [3]:

$\alpha = -1\frac{1}{2}, \beta = \frac{1}{3}$:

$$\left(-1\tfrac{1}{2}\right)^2\left(\frac{1}{3}\right) = \frac{3}{4}$$

$$= \text{RHS}$$

$\alpha = -\frac{5}{18}, \beta = -2\frac{1}{9}$:

$$\left(-\frac{5}{18}\right)^2\left(-2\tfrac{1}{9}\right) = -\frac{475}{2916}$$

$$\neq \text{RHS}$$

$\therefore$ The roots are $-1\frac{1}{2}$ and $\frac{1}{3}$.

Quartic equations

If a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots α, β, γ and δ then:

$$(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) = 0$$

When expanded fully, this is:

$$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta = 0$$

The quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ can be written in monic form as:

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0.$$

$$\therefore\ x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta$$

$$\equiv x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a}$$

Equating coefficients gives the formulas below:

Sum and product of the roots of a quartic equation

For the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$:

Sum of roots: $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$

Sum of roots 2 at a time: $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$

Sum of roots 3 at a time: $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$

Product of roots: $\alpha\beta\gamma\delta = \frac{e}{a}$

Do you notice a pattern in these formulas?

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INVESTIGATION

HIGHER DEGREE POLYNOMIALS

This pattern of roots and coefficients extends to polynomials of any degree.

Can you find results for sums and products of roots for polynomial equations of degree 5, 6 and so on?

EXAMPLE 10

If α, β, γ and δ are the roots of $x^4 - 2x^3 + 7x - 3 = 0$, find:

a $\alpha\beta\gamma\delta$ **b** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

Solution

a
$$\begin{aligned}\alpha\beta\gamma\delta &= \frac{e}{a}\\ &= \frac{-3}{1}\\ &= -3\end{aligned}$$

b
$$\begin{aligned}\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= -\frac{d}{a}\\ &= -\frac{7}{1}\\ &= -7\end{aligned}$$

c
$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} &= \frac{\beta\gamma\delta}{\alpha\beta\gamma\delta} + \frac{\alpha\gamma\delta}{\alpha\beta\gamma\delta} + \frac{\alpha\beta\delta}{\alpha\beta\gamma\delta} + \frac{\alpha\beta\gamma}{\alpha\beta\gamma\delta}\\ &= \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}\\ &= \frac{-7}{-3}\\ &= 2\tfrac{1}{3}\end{aligned}$$

EXT1 Exercise 6.04 Roots and coefficients of polynomial equations

1 Given that α and β are the roots of the equation, for each quadratic equation find:

i $\alpha + \beta$ **ii** $\alpha\beta$

a $x^2 - 2x + 8 = 0$ **b** $3x^2 + 6x - 2 = 0$ **c** $x^2 + 7x + 1 = 0$

d $4x^2 - 9x - 12 = 0$ **e** $5x^2 + 15x = 0$

2 Where α, β, and γ are the roots of the equation, for each cubic equation find:

i $\alpha + \beta + \gamma$ **ii** $\alpha\beta + \alpha\gamma + \beta\gamma$ **iii** $\alpha\beta\gamma$

a $x^3 + x^2 - 2x + 8 = 0$ **b** $x^3 - 3x^2 + 5x - 2 = 0$ **c** $2x^3 - x^2 + 6x + 2 = 0$

d $-x^3 - 3x^2 - 11 = 0$ **e** $x^3 + 7x - 3 = 0$

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3 For each quartic equation, where α, β, γ and δ are the roots of the equation, find:

i $\alpha + \beta + \gamma + \delta$

ii $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

iii $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

iv $\alpha\beta\gamma\delta$

a $x^4 + 2x^3 - x^2 - x + 5 = 0$

b $x^4 - x^3 - 3x^2 + 2x - 7 = 0$

c $-x^4 + x^3 + 3x^2 - 2x + 4 = 0$

d $2x^4 - 2x^3 - 4x^2 + 3x - 2 = 0$

e $2x^4 - 12x^3 + 7 = 0$

4 If α and β are the roots of $x^2 - 5x - 5 = 0$, find:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta}$ **d** $\alpha^2 + \beta^2$

5 If α, β and γ are the roots of $2x^3 + 5x^2 - x - 3 = 0$, find:

a $\alpha\beta\gamma$ **b** $\alpha\beta + \alpha\gamma + \beta\gamma$ **c** $\alpha + \beta + \gamma$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **e** $(\alpha + 1)(\beta + 1)(\gamma + 1)$

6 If α, β, γ and δ are the roots of $x^4 - 2x^3 + 5x - 3 = 0$, find:

a $\alpha\beta\gamma\delta$ **b** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

7 One root of $x^2 - 3x + k - 2 = 0$ is -4. Find the value of k.

8 One root of $x^3 - 5x^2 - x + 21 = 0$ is 3. Find the sum $\alpha + \beta$ and the product $\alpha\beta$ of the other 2 roots.

9 Given $P(x) = 2x^3 - 7x^2 + 4x + 1$, if the equation $P(x) = 0$ has zero $x = 1$, find the sum and product of its other roots.

10 Find the value(s) of k if the quadratic equation $x^2 - (k + 2)x + k + 1 = 0$ has:

a equal roots **b** one root equal to 5

c consecutive roots **d** one root double the other

e reciprocal roots.

11 Two roots of $x^3 + ax^2 + bx + 24 = 0$ are equal to 4 and -2. Find the values of a and b.

12 **a** Show that 1 is a zero of the polynomial $P(x) = x^4 - 2x^3 + 7x - 6$.

b If α, β and γ are the other 3 zeros, find the value of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

13 If $x = 2$ is a double root of $ax^4 - 2x^3 - 8x + 16 = 0$, find the value of a and the sum of the other 2 roots.

14 Two of the roots of $x^3 - px^2 - qx + 30 = 0$ are 3 and 5.

a Find the other root. **b** Find p and q.

15 The product of two of the roots of $x^4 + 2x^3 - 18x - 5 = 0$ is -5 Find the product of the other 2 roots.

16 The sum of 2 of the roots of $x^4 + x^3 + 7x^2 + 14x - 1 = 0$ is 4. Find the sum of the other 2 roots.

17 Find the roots of $x^3 - 3x^2 + 4 = 0$ given that 2 of the roots are equal.

18 Solve $12x^3 - 4x^2 - 3x + 1 = 0$ if the sum of 2 of its roots is 0.

19 Solve $6x^4 + 5x^3 - 24x^2 - 15x + 18 = 0$ if the sum of 2 of its roots is zero.

20 Two roots of $x^3 + mx^2 - 3x - 18 = 0$ are equal and rational. Find m.

WS Polynomial graphs

Graphing polynomials

WS Sketching curves

EXT1 6.05 Graphing polynomial functions

To graph polynomial functions, factorise polynomials to find their zeros first.

EXAMPLE 11

a Factorise the polynomial $P(x) = x^3 - x^2 - 5x - 3$.

b Sketch the graph of the polynomial.

Solution

a Factors of -3 are ± 1 and ± 3.

$P(-1) = (-1)^3 - (-1)^2 - 5(-1) - 3$

$= 0$

So $x + 1$ is a factor of the polynomial.

By long division,

$$\begin{array}{r} x^2 - 2x - 3 \\ x+1 \overline{)\, x^3 - x^2 - 5x - 3} \\ \underline{x^3 + x^2 } \\ -2x^2 - 5x \\ \underline{-2x^2 - 2x} \\ -3x - 3 \\ \underline{-3x - 3} \\ 0 \end{array}$$

$P(x) = (x + 1)(x^2 - 2x - 3)$

$= (x + 1)(x - 3)(x + 1)$

$= (x + 1)^2(x - 3)$

b For the graph of $P(x) = x^3 - x^2 - 5x - 3$,

For x-intercepts, $P(x) = 0$:

$0 = x^3 - x^2 - 5x - 3$

$= (x + 1)^2(x - 3)$

$x = -1, 3$

So x-intercepts are -1 and 3.

For y-intercept, $x = 0$:

$P(0) = 0^3 - (0)^2 - 5(0) - 3$

$= -3$

So y-intercept is -3.

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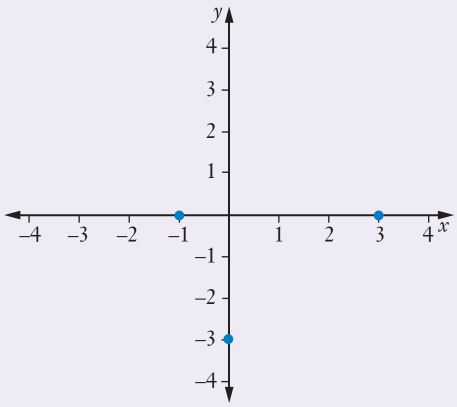


Test $x < -1$, say $x = -2$:

$$\begin{aligned} P(-2) &= (-2+1)^2(-2-3) \\ &= (-1)^2(-5) \\ &= -5 \\ &< 0 \end{aligned}$$

So the curve is below the x-axis for $x < -1$.

Test $-1 < x < 3$, say $x = 0$:

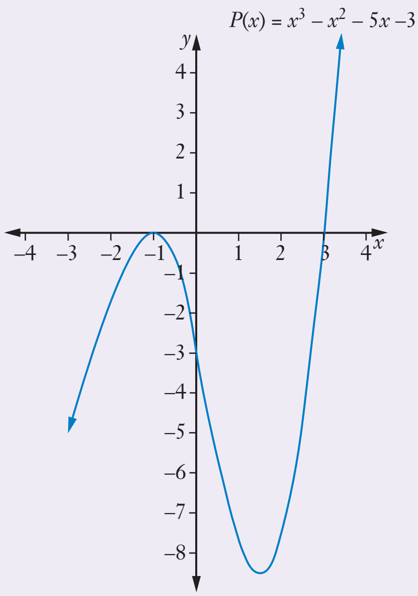
$$\begin{aligned} P(0) &= (0+1)^2(0-3) \\ &= (1)^2(-3) \\ &= -3 \\ &< 0 \end{aligned}$$

So the curve is below the x-axis for $-1 < x < 3$.

Test $x > 3$, say $x = 4$:

$$\begin{aligned} P(4) &= (4+1)^2(4-3) \\ &= (5)^2(1) \\ &= 25 \\ &> 0 \end{aligned}$$

So the curve is above the x-axis for $x > 3$.



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Limiting behaviour of polynomials

What does the graph of a polynomial look like for large positive and negative values of x as $x \to \pm\infty$?

INVESTIGATION

LIMITING BEHAVIOUR OF POLYNOMIALS

Use a graphics calculator or graphing software to explore the behaviour of polynomials as x becomes large (both negative and positive values).

For example, sketch $f(x) = 2x^5 + 3x^2 - 7x - 1$ and $f(x) = 2x^5$ together. What do you notice at both ends of the graphs where x is large? Zoom out on these graphs and watch the graph of the polynomial and the graph of the leading term come together.

Try sketching the graphs of other polynomials along with graphs of their leading terms. Do you find the same results?

The leading term, a_nx^n of a polynomial function shows us what the limiting behaviour of the function will be.

For very large $|x|$, $P(x) \approx a_nx^n$.

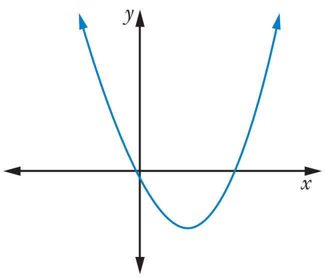
As x becomes large, the leading term a_nx^n becomes very large compared with the other terms because it has the highest power of x and the other powers of x are relatively small.

Consider a polynomial of even degree; for example, a polynomial whose leading term a_nx^n is $3x^2$ or x^4 or $-5x^6$.

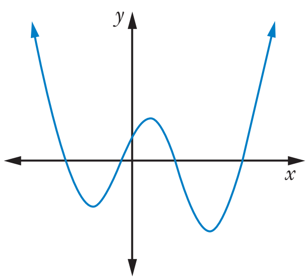
If n is even, x^n is always positive.

So if n is even and $a_n > 0$, then $a_nx^n > 0$.

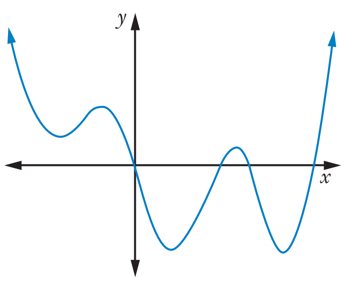
As $x \to \pm\infty$, $P(x) \to \infty$, as shown by these 3 graphs of polynomials.



Degree 2 (quadratic)



Degree 4 (quartic)



Degree 6

If n is even and $a_n < 0$, then $a_nx^n < 0$.

As $x \to \pm\infty$, $P(x) \to -\infty$, as shown by the 3 graphs of polynomials on the next page.

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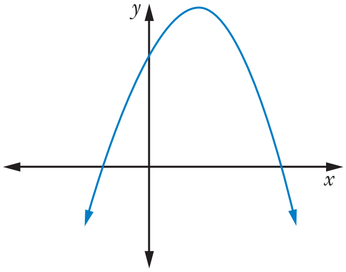
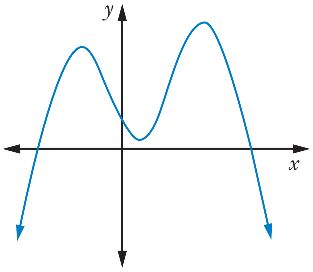
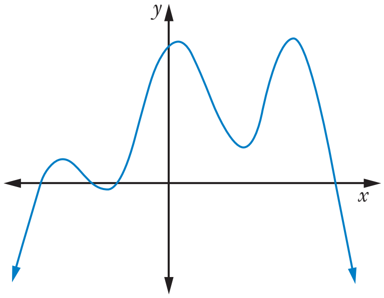


Degree 2 (quadratic)



Degree 4 (quartic)



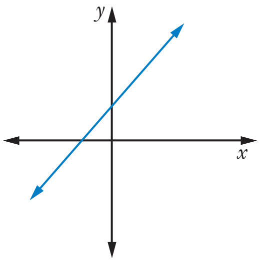
Degree 6

Now consider a polynomial of odd degree; for example, a polynomial whose leading term $a_n x^n$ is x^3 or $-4x^5$ or $2x^7$.

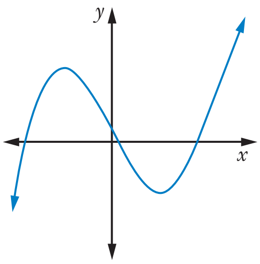
If n is odd, $x^n > 0$ for $x > 0$ and $x^n < 0$ for $x < 0$.

So if n is odd and a_n is positive, then $a_n x^n > 0$ for $x > 0$ and $a_n x^n < 0$ for $x < 0$.

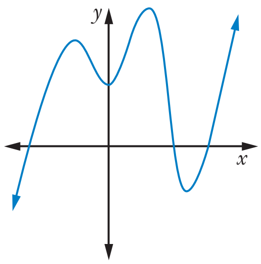
As $x \to -\infty$, $P(x) \to -\infty$ and as $x \to \infty$, $P(x) \to \infty$, as shown by these 3 graphs of polynomials.



Degree 1 (linear)



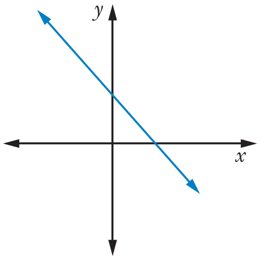
Degree 3 (cubic)



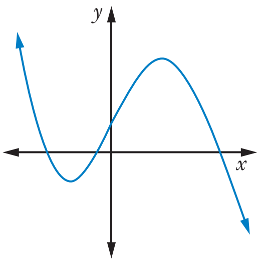
Degree 5

If n is odd and a_n is negative, then $a_n x^n < 0$ for $x > 0$ and $a_n x^n > 0$ for $x < 0$.

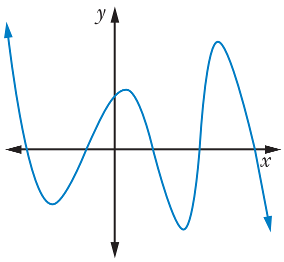
As $x \to -\infty$, $P(x) \to \infty$ and as $x \to \infty$, $P(x) \to -\infty$, as shown by these 3 graphs of polynomials.



Degree 1 (linear)



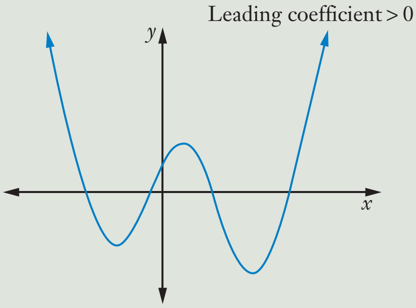
Degree 3 (cubic)

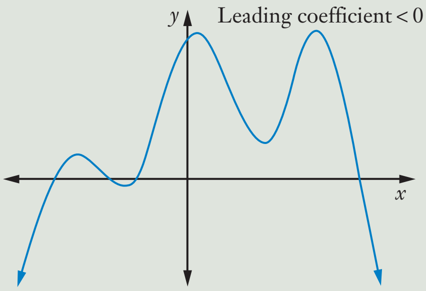


Degree 5

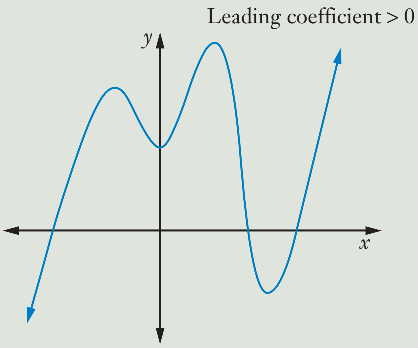
The graph of a polynomial

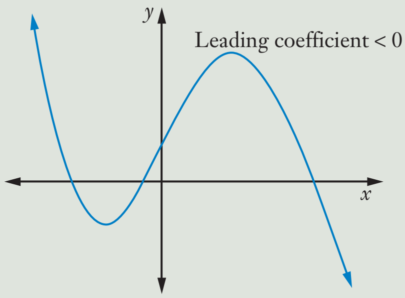
If $P(x)$ has **even degree**, the ends of the graph both point in the same direction.





If $P(x)$ has **odd degree**, the ends of the graph point in opposite directions.





EXT1 Exercise 6.05 Graphing polynomial functions

1 **a** Show that $x - 2$ is a factor of $P(x) = x^3 - 3x^2 - 4x + 12$.

b Write $P(x)$ as a product of its factors.

c Sketch the graph of the polynomial.

2 Sketch the graph of each polynomial, showing all x- and y-intercepts.

a $P(x) = x^3 + 3x^2 - 10x - 24$

b $P(x) = x^3 + x^2 - 9x - 9$

c $P(x) = 12 - 19x + 8x^2 - x^3$

d $P(x) = x^3 - 13x + 12$

e $P(x) = -x^3 + 2x^2 + 9x - 18$

f $P(x) = x^3 + 2x^2 - 4x - 8$

g $P(x) = x^3 - 5x^2 + 8x - 4$

h $P(x) = x^3 + x^2 - 5x + 3$

i $P(x) = 16x + 12x^2 - x^4$

j $P(x) = x^4 - 2x^2 + 1$

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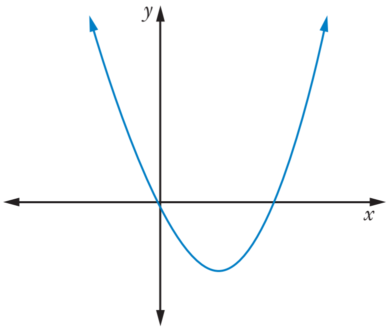
3 For each graph, state if:

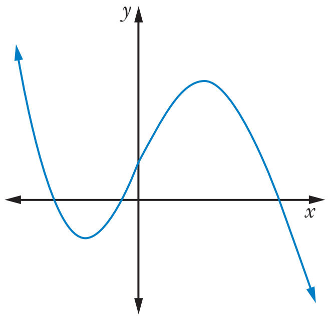
i the leading coefficient is positive or negative

ii the degree of the polynomial is even or odd.

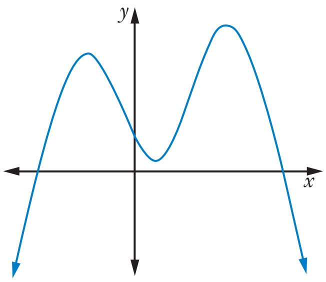
a



b



c



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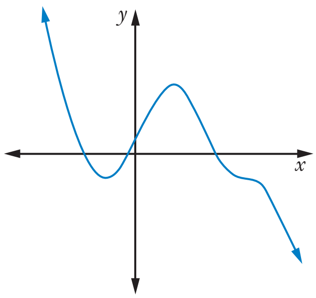
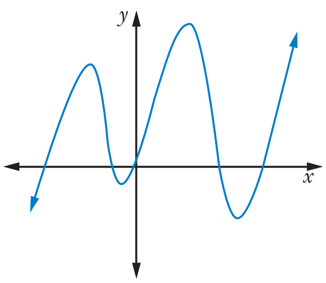
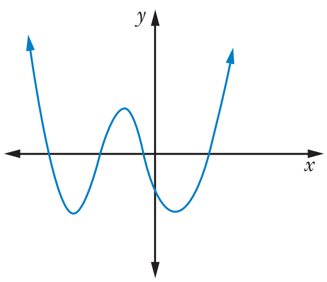


e



f



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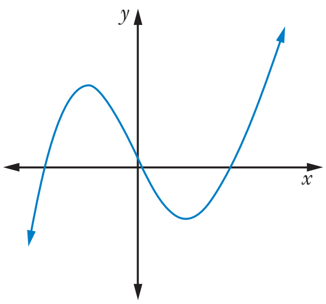

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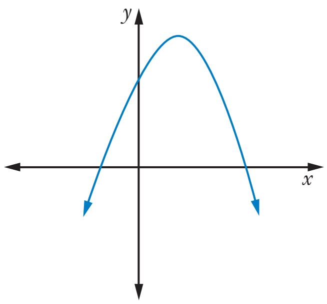

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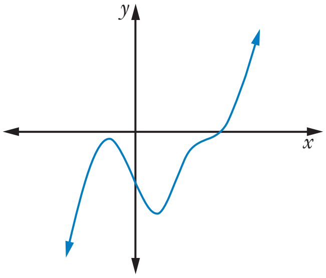
i



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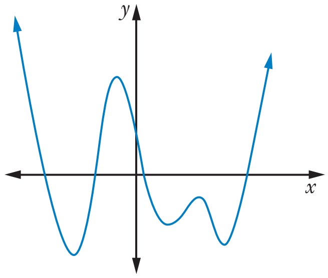


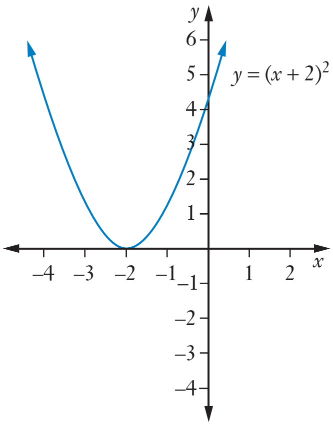
4 Draw an example of a polynomial with leading term:

a x^3 **b** $-2x^5$ **c** $3x^2$

d $-x^4$ **e** $-2x^3$

EXT1 6.06 Multiple roots

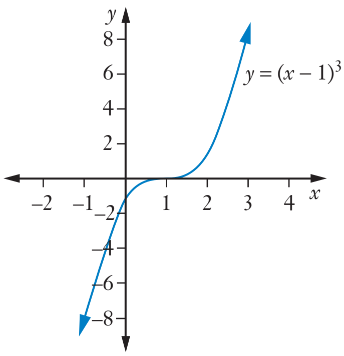
If $f(x) = (x + 2)^2$, we say that the quadratic equation $f(x) = 0$ has a double root at $x = -2$ since there are 2 equal roots.



Double root at $x = -2$
Turning point at $x = -2$

Similarly, if $f(x) = (x - 1)^3$, the cubic equation $f(x) = 0$ has a triple root at $x = 1$.

Notice that there is always a turning point or point of inflection where there is a multiple root.



Triple root at $x = 1$
Point of inflection at $x = 1$

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INVESTIGATION

MULTIPLE ROOTS

Use a graphics calculator or graphing software to graph polynomials with multiple roots.

a Examine values close to the roots.

b Look at the relationship between the degree of the polynomial, the leading coefficient and its graph.

1 $P(x) = (x + 1)(x - 3)$

2 $P(x) = (x + 1)^2(x - 3)$

3 $P(x) = -(x + 1)^3(x - 3)$

4 $P(x) = -(x + 1)^4(x - 3)$

5 $P(x) = (x + 1)(x - 3)^2$

6 $P(x) = (x + 1)(x - 3)^3$

7 $P(x) = -(x + 1)(x - 3)^4$

8 $P(x) = -(x + 1)^2(x - 3)^2$

9 $P(x) = -(x + 1)^2(x - 3)^3$

10 $P(x) = (x + 1)^3(x - 3)^2$

Multiple roots

If $P(x) = (x - k)^2Q(x)$ then $P(x) = 0$ has a **double root** at $x = k$ (2 equal roots)

If $P(x) = (x - k)^3Q(x)$ then $P(x) = 0$ has a **triple root** at $x = k$ (3 equal roots)

If $P(x) = (x - k)^rQ(x)$ then $P(x) = 0$ has a **multiple root** at $x = k$ (r equal roots).

We can also say that $P(x)$ has a root with **multiplicity** r at $x = k$.

EXAMPLE 12

a Examine the behaviour of the polynomial $P(x) = (x + 2)^2(x - 1)$ close to its multiple root and describe how this affects its graph at this root.

b Describe the limiting behaviour of the polynomial.

c Sketch the graph of the polynomial.

Solution

a $P(x) = (x + 2)^2(x - 1)$ has a double root at $x = -2$.

Look at the sign of $P(x)$ close to $x = -2$:

On LHS:

$$P(-2.1) = (-2.1 + 2)^2(-2.1 - 1) = -0.031 < 0$$

So the curve is below the x-axis on the LHS.

On RHS:

$$P(-1.9) = (-1.9 + 2)^2(-1.9 - 1) = -0.029 < 0$$

So the curve is also below the x-axis on the RHS.

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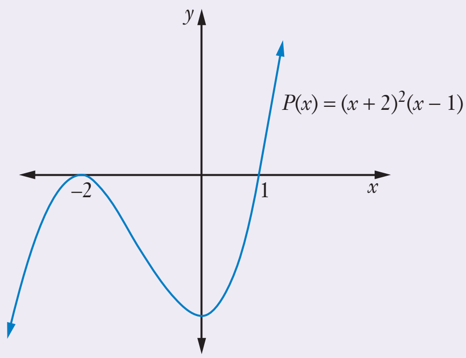
As $P(x)$ is negative on both sides of the double root at $x = -2$, its graph is below the x-axis. But $x = -2$ is an x-intercept, so there is a maximum turning point at that point.

b $P(x) = (x + 2)^2(x - 1)$ has the leading term x^3 so it has an odd degree (3) and a positive leading coefficient (1). As $x \to \infty$, $P(x) \to \infty$. Since $P(x)$ is odd, as $x \to -\infty$, $P(x) \to -\infty$.

c



Turning points at multiple roots on polynomial graphs

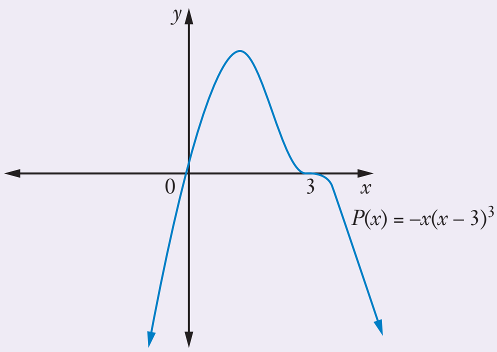
If the multiplicity r of a root is even, there is a maximum or minimum turning point at the multiple root.

If the multiplicity r of a root is odd, there is a point of inflection at the multiple root.

EXAMPLE 13

Sketch the graph of $P(x) = -x(x - 3)^3$.

Solution



$P(x) = -x(x - 3)^3 = 0$ has roots at $x = 0$, $x = 3$.

$x = 0$ is a single root so the curve crosses the x-axis at this point.

$x = 3$ is a triple root. Since r is odd, there is a point of inflection at $x = 3$.

$P(x) = -x(x - 3)^3$ has leading term $-x^4$, so $P(x)$ has an even degree and a negative leading coefficient.

As $x \to \pm\infty$, $P(x) \to -\infty$.

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EXT1 Exercise 6.06 Multiple roots

1 Find the roots of each polynomial equation $P(x) = 0$ and state if they are multiple roots.

a $P(x) = x^2 - 6x + 9$

b $P(x) = x^3 - 9x^2 + 14x$

c $P(x) = x^3 - 3x^2$

d $f(x) = x^3 - 2x^2 - 4x + 8$

e $P(x) = x^3 - 6x^2 + 12x - 8$

f $A(x) = x^4 - 4x^3 + 5x^2 - 2x$

g $P(x) = x^4 - 4x^3 - 2x^2 + 12x + 9$

h $Q(x) = x^5 - 8x^4 + 16x^3$

i $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

j $f(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$

2 A monic polynomial of degree 2 has a double root at $x = -4$. Write down an expression for the polynomial $P(x)$. Is this a unique expression?

3 A polynomial of degree 3 has a triple root at $x = 1$.

a Write down an expression for the polynomial. Is this unique?

b If $P(2) = 5$, write the expression for the polynomial.

4 Sketch the graph of a polynomial with a double root at $x = 2$ and leading term $2x^3$.

5 Sketch the graph of a polynomial with a double root at $x = -1$ and leading term $-x^3$.

6 Sketch the graph of a polynomial with a double root at $x = 2$ and a leading term x^4.

7 Sketch the graph of a polynomial with a double root at $x = -3$ and leading term x^6.

8 A polynomial has a triple root at $x = 1$ and leading term x^3. Sketch a graph showing this information.

9 Given a polynomial with a triple root at $x = 0$ and leading term $-x^4$, sketch the graph of a polynomial that fits this information.

10 If a polynomial has a triple root at $x = -2$ and a leading term of x^8, sketch the graph of a polynomial fitting this information.

11 A polynomial has a triple root at $x = 4$ and its leading term is $-4x^3$. Sketch its graph.

12 A monic polynomial has degree 3 and a double root at $x = -1$. Show on a sketch that the polynomial has another root.

13 A polynomial with leading term $-x^8$ has a triple root at $x = -2$. Show by a sketch that the polynomial has at least one other root.

14 A polynomial has a double root at $x = 2$ and a double root at $x = -3$. Its leading term is $2x^5$. By sketching a graph, show that the polynomial has another root.

EXT1 6.07 The inverse of a function

The inverse of a function is an operation that 'undoes' the original function.
For example:

- The inverse relation of $y = 2x$ is $y = \frac{x}{2}$.
- The inverse relation of $y = \sqrt{x}$ is $y = x^2$.

EXAMPLE 14

Change the subject of each function to x, and then find the inverse relation of the function.

a $y = 2x + 1$ **b** $y = x^3 - 2$

Solution

a Make x the subject of the function:

$$y = 2x + 1$$

$$y - 1 = 2x$$

$$\frac{y-1}{2} = x$$

The inverse operations of 'multiplying by 2 then adding 1' are 'subtracting 1 then dividing by 2'.

So the inverse relation of $y = 2x + 1$ is $y = \frac{x-1}{2}$.

b

$$y = x^3 - 2$$

$$y + 2 = x^3$$

$$\sqrt[3]{y+2} = x$$

The inverse operations of 'cubing then subtracting 2' are 'adding 2 then finding the cube root'.

So the inverse relation of $y = x^3 - 2$ is $y = \sqrt[3]{x+2}$.

Notice in the example that for the inverse relation, we can swap x and y.

Finding the inverse relation of a function

The inverse relation of $y = f(x)$ can be found by interchanging the x and y of the function, then making y the subject.

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EXAMPLE 15

Find the inverse relation of:

a $y = 3x - 8$ **b** $f(x) = 2x^5 + 7$ **c** $y = x^2 + 4x - 7$

Solution

a
$$\begin{aligned} x &= 3y - 8 \\ x + 8 &= 3y \\ \frac{x+8}{3} &= y \end{aligned}$$

b
$$\begin{aligned} x &= 2y^5 + 7 \\ x - 7 &= 2y^5 \\ \frac{x-7}{2} &= y^5 \\ \sqrt[5]{\frac{x-7}{2}} &= y \end{aligned}$$

c
$$\begin{aligned} x &= y^2 + 4y - 7 \\ x + 7 &= y^2 + 4y \\ x + 7 + 4 &= y^2 + 4y + 4 \\ x + 11 &= (y+2)^2 \\ \pm\sqrt{x+11} &= y + 2 \\ \pm\sqrt{x+11} - 2 &= y \end{aligned}$$

Completing the square

EXT1 Exercise 6.07 The inverse of a function

1 Find the inverse relation of each function.

a $y = 3x$ **b** $y = -x$ **c** $f(x) = \frac{x}{5}$
d $y = \sqrt[3]{x}$ **e** $y = 7x$ **f** $f(x) = x + 1$
g $y = x - 5$ **h** $f(x) = x + 3$ **i** $y = x^3$
j $y = x^5$ **k** $f(x) = x - 9$ **l** $f(x) = 5 - x$
m $y = -3x$ **n** $y = x^2$ **o** $y = \sqrt[7]{x}$
p $y = \frac{x}{9}$ **q** $y = x^8$

2 Find the inverse relation of each function.

a $y = x^3 + 5$ **b** $f(x) = x^7 - 1$ **c** $y = \sqrt[3]{x-2}$
d $y = \frac{2}{x}$ **e** $y = \frac{3}{x+5}$ **f** $y = \frac{x+1}{2}$
g $f(x) = \sqrt{x+2}$ **h** $y = \sqrt[3]{x-7}$ **i** $y = \frac{3}{\sqrt{x}}$
j $y = 3x^5 - 2$ **k** $f(x) = 2\sqrt{x} + 5$ **l** $y = 3\sqrt[3]{2x+1}$
m $y = 2x^4$ **n** $y = x^2 + 5$ **o** $y = x^6 - 3$
p $y = x^2 + 8x$ **q** $y = 4x - x^2$ **r** $y = x^2 - 2x + 3$
s $y = x^2 + 10x - 1$ **t** $y = x^2 - 6x - 3$ **u** $y = x^2 + 12x - 11$

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EXT1 6.08 Graphing the inverse of a function

Graph of the inverse of a function

On the number plane, the graph of the inverse relation is a reflection of the graph of the original function in the line $y = x$.

If a point (x, y) on the number plane has its x- and y-coordinates swapped, then the point (y, x) is the reflection of (x, y) in the line $y = x$.

EXAMPLE 16

Sketch the graph of the original function, its inverse and the line $y = x$ on the same set of axes.

a $y = x + 3$ **b** $y = x^3$ **c** $y = x^2$

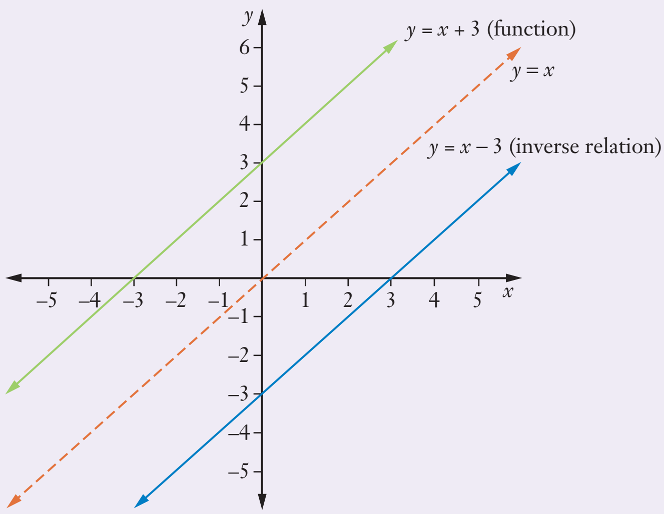
Solution

a $y = x + 3$ is a line with gradient 1 and y-intercept 3.

For x-intercept, $y = 0$:
$0 = x + 3$
$x = -3$

Inverse of $y = x + 3$:
$x = y + 3$
$y = x - 3$
This is a line with gradient 1 and y-intercept -3.

For x-intercept, $y = 0$:
$0 = x - 3$
$x = 3$



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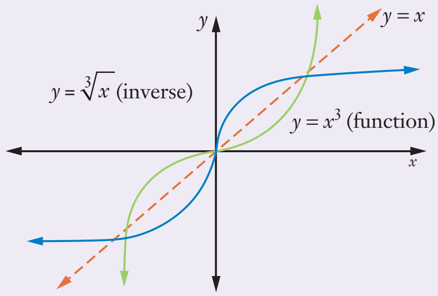
b $y = x^3$ is a cubic function with a point of inflection at (0, 0).

Inverse of $y = x^3$:

$$x = y^3$$

$$y = \sqrt[3]{x}$$

x	−2	−1	0	1	2
y	−1.26	−1	0	1	1.26



c $y = x^2$ is a quadratic function with a turning point at (0, 0).

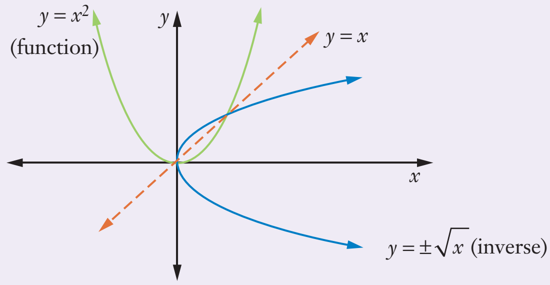
Inverse of $y = x^2$:

$$x = y^2$$

$$y = \pm\sqrt{x}$$

x	0	1	4	9
y	0	±1	±2	±9

Notice that $y = \pm\sqrt{x}$ is not a function.



The inverse relations of $y = x + 3$ and $y = x^3$ are also functions, while the inverse relation of $y = x^2$ is not. How could you test the original function to see if its inverse is a function?

Horizontal line test

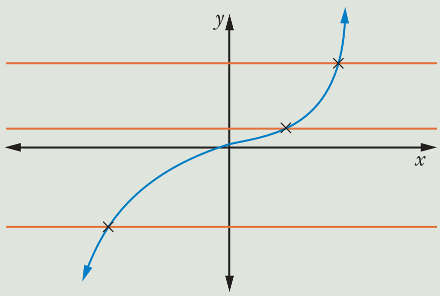
A function has a unique y value for every x value. This can be determined by a vertical line test.

Since the inverse is an exchange of the x and y values, the **inverse function** exists if there is a unique value of x for every y value in the original function, that is, if the original function is **one-to-one**. As we saw in Chapter 4, this can be determined by a **horizontal line test**.

Horizontal line test

If any horizontal line crosses the graph of a function at only one point, then the inverse relation is a function.

This also means that the original function is a **one-to-one function**.



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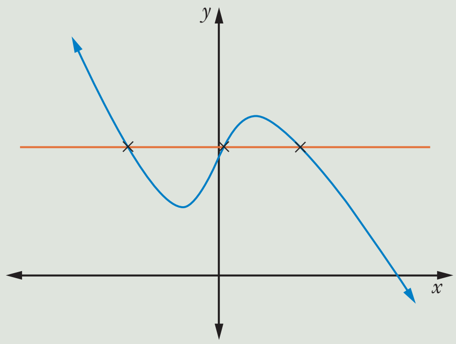
If a horizontal line crosses the graph at more than one point, then the inverse relation is not a function.



Notice that functions that pass the horizontal line test are either always increasing or always decreasing. They do not have turning points. We call these functions **monotonic increasing** or **monotonic decreasing**.

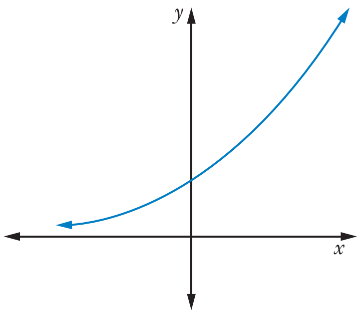
EXT1 Exercise 6.08 Graphing the inverse of a function

1 Sketch the graph of each function, its inverse and the line $y = x$ on the same set of axes.

a $f(x) = 2x + 1$ **b** $y = x^3 - 1$ **c** $f(x) = \frac{x}{4}$ **d** $y = \sqrt{x+1}$

2 Does the function represented by each graph have an inverse function?

a



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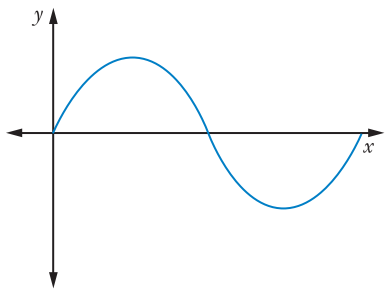

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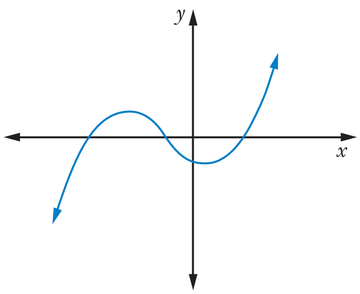

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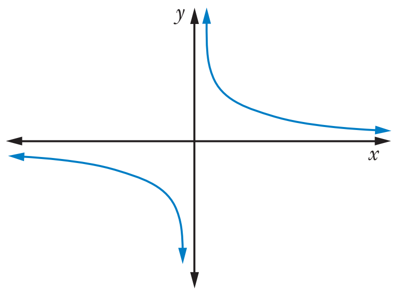

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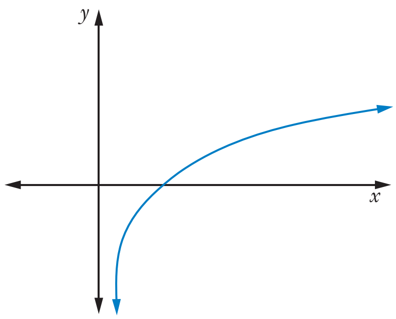

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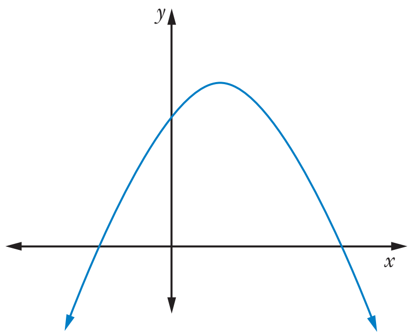

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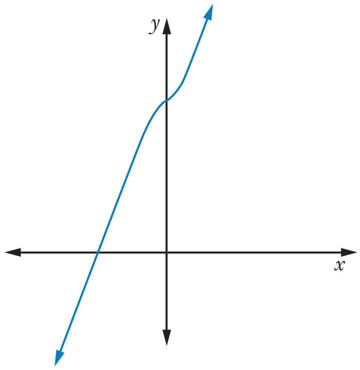

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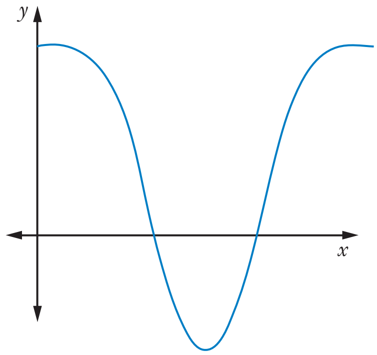

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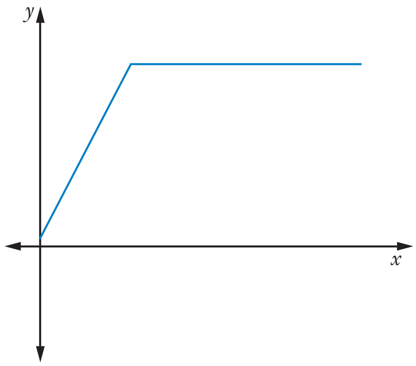

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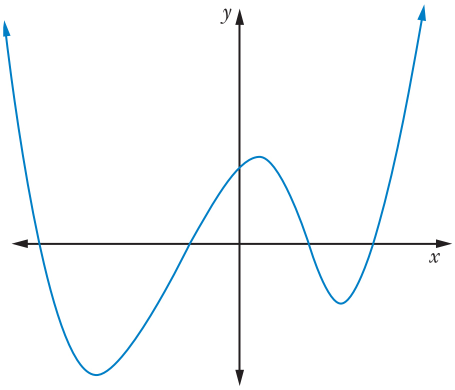

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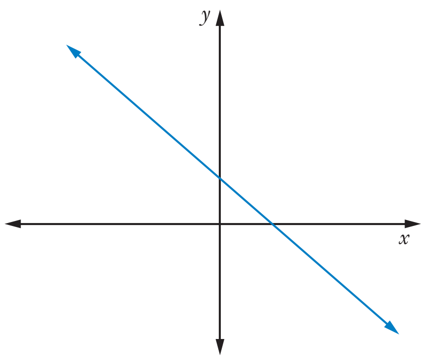

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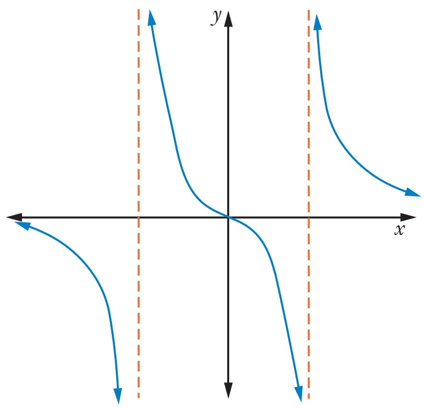

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Inverse functions

Inverse functions code puzzle

EXT1 6.09 Inverse functions

Inverse function notation

If the original function is $y = f(x)$, then we write the inverse function as $y = f^{-1}(x)$.

Note: $f^{-1}(x)$ is not the same as the reciprocal function $[f(x)]^{-1} = \dfrac{1}{f(x)}$.

Because x and y are interchanged in inverse functions, the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function.

Domain and range of inverse functions

If $y = f(x)$ is a one-to-one function with domain $[a, b]$ and range $[f(a), f(b)]$, the inverse function $y = f^{-1}(x)$ has domain $[f(a), f(b)]$ and range $[a, b]$.

Inverse functions

EXAMPLE 17

a Find the domain and range of the function $y = \dfrac{1}{x-2}$.

b Find the inverse function.

c Find the domain and range of the inverse function.

Solution

a The denominator can't be 0.

$x - 2 \neq 0$

$x \neq 2$

Domain: $(-\infty, 2) \cup (2, \infty)$

$\dfrac{1}{x-2} \neq 0$

So $y \neq 0$

Range: $(-\infty, 0) \cup (0, \infty)$

b

$$x = \frac{1}{y-2}$$

$$y - 2 = \frac{1}{x}$$

$$y = \frac{1}{x} + 2$$

c Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 2) \cup (2, \infty)$

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Restricting the domain

If a function fails the horizontal line test and is not one-to-one, we can still create an inverse function if we restrict its domain to where it is monotonic increasing or decreasing only (no turning points). Then it will have an inverse function over that **restricted domain**.

EXAMPLE 18

Restrict the domain of each function to find an inverse function and its domain and range.

a $y = x^2$

b $f(x) = x^2 - 4x$

Solution

a The inverse relation is

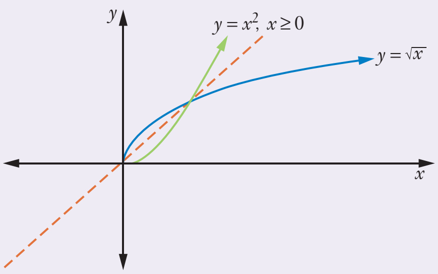
$$x = y^2$$
$$y = \pm\sqrt{x}$$

$y = x^2$ is a parabola with a minimum turning point at (0, 0).

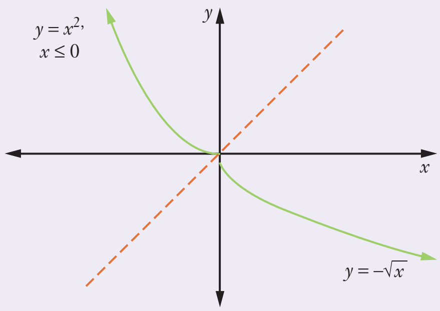
We can restrict its domain to where it is monotonic increasing in the interval $x \geq 0$.

So $f(x)$ has domain $[0, \infty)$ and range $[0, \infty)$, and $f^{-1}(x)$ must have domain $[0, \infty)$ and range $[0, \infty)$.

$\therefore$ The inverse function is $y = \sqrt{x}$.



Alternatively, if the domain of $y = x^2$ is restricted to $x \leq 0$ where it is monotonic decreasing, then the inverse function is $y = -\sqrt{x}$ with domain $[0, \infty)$ and range $(-\infty, 0]$.



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b Inverse relation:

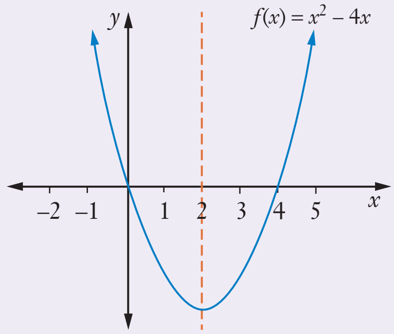
$$x = y^2 - 4y$$

$$x + 4 = y^2 - 4y + 4$$ Completing the square

$$= (y - 2)^2$$

$$\pm\sqrt{x+4} = y - 2$$

$$\therefore y = \pm\sqrt{x+4} + 2$$



$f(x) = x^2 - 4x$ is a concave upwards parabola with x-intercepts 0, 4 and axis of symmetry at $x = 2$.

$$f(2) = 2^2 - 4(2)$$

$$= -4$$

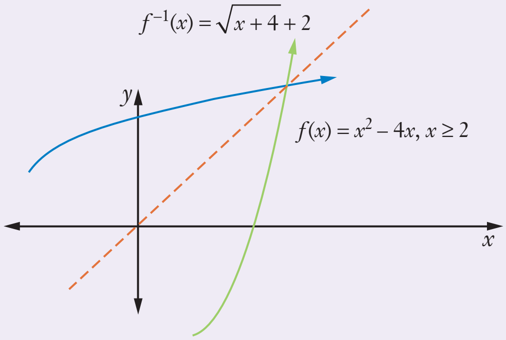
Minimum turning point at $(2, -4)$.

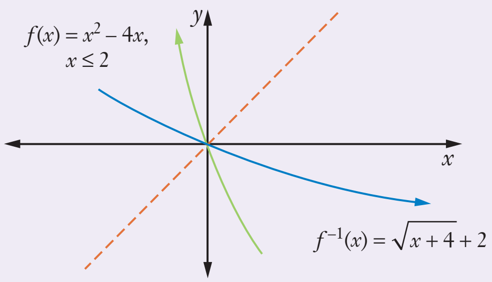
$f(x)$ is monotonic increasing for $x \geq 2$. If the domain is restricted to $[2, \infty)$, the range will be $[-4, \infty)$.

$f^{-1}(x)$ will have domain $[-4, \infty)$ and range $[2, \infty)$.

$$\therefore f^{-1}(x) = \sqrt{x+4} + 2$$

Similarly, if the domain of $f(x)$ is restricted to $(-\infty, 2]$ the inverse function will be $f^{-1}(x) = -\sqrt{x+4} + 2$ with domain $[-4, \infty)$ and range $(-\infty, 2]$.





A function and its inverse 'undo each other'. That is, $f^{-1}[f(x)] = f[f^{-1}(x)] = x$.

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EXAMPLE 19

If $y = 2x - 5$, find the inverse function and show that $f^{-1}[f(x)] = f[f^{-1}(x)] = x$.

Solution

$$\begin{aligned}
x &= 2y - 5 \\
x + 5 &= 2y \\
\frac{x+5}{2} &= y \\
f^{-1}(x) &= \frac{x+5}{2} \\
f^{-1}[f(x)] &= f^{-1}(2x - 5) \\
&= \frac{(2x-5)+5}{2} \\
&= \frac{2x}{2} \\
&= x
\end{aligned}$$

$$\begin{aligned}
f[f^{-1}(x)] &= f\left(\frac{x+5}{2}\right) \\
&= 2\left(\frac{x+5}{2}\right) - 5 \\
&= x + 5 - 5 \\
&= x
\end{aligned}$$

$$\therefore f^{-1}[f(x)] = f[f^{-1}(x)] = x$$

EXT1 Exercise 6.09 Inverse functions

1 Which of these functions has an inverse function? There is more than one answer.

A $f(x) = 5x - 7$ **B** $y = \dfrac{4}{x}$ **C** $y = x^2 + 1$ **D** $y = x^3$

2 Find the inverse function of each function, and state its domain and range.

a $y = x^3$ **b** $y = 3x - 2$ **c** $f(x) = \dfrac{2}{x}$ **d** $y = \dfrac{1}{x+1}$

3 If the domain of each function is restricted to a monotonic increasing curve, find the inverse function and its domain and range.

a $y = 2x^2$ **b** $y = x^2 + 2$ **c** $y = (x - 3)^2$

d $y = x^2 - 2x$ **e** $y = x^6$ **f** $y = 1 - x^2$

g $y = x^4 - 1$ **h** $y = \dfrac{1}{x^2}$

4 **a** Find the domain over which the function $y = x^2 + 6x$ is monotonic increasing.

b Find the inverse function over this restricted domain, and state its domain and range.

c Find the domain over which $y = x^2 + 6x$ is monotonic decreasing.

d Find the inverse function over this restricted domain, and state its domain and range.

5 Restrict the domain of each function to a monotonic decreasing curve and find the inverse function over this domain.

a $y = x^2$ **b** $y = 3x^2 - 1$ **c** $f(x) = (x-2)^4$

d $y = \dfrac{3}{x^2}$ **e** $f(x) = \dfrac{2}{x^4}$

6 For each function and its inverse, show that $f^{-1}[f(x)] = f[f^{-1}(x)] = x$.

a $f(x) = x + 7$ **b** $y = 3x$ **c** $y = \sqrt{x}$

d $y = 3x + 1$

7 a Find the domain and range of $y = \dfrac{2}{x-1}$.

b Find the inverse function.

c State the domain and range of the inverse function.

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6. TEST YOURSELF

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

Polynomials review

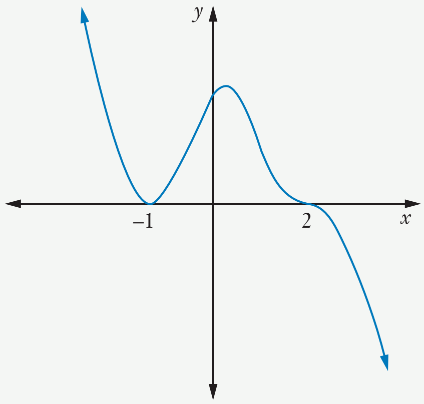
1 Which is a possible equation for this graph?

A $P(x) = (x + 1)^2(x - 2)$

B $P(x) = (x - 1)(x + 2)^2$

C $P(x) = -(x + 1)^2(x - 2)^3$

D $P(x) = -(x - 1)^2(x + 2)^3$



2 If $f(x) = \dfrac{1}{x-3}$, find $f^{-1}(x)$:

A $f^{-1}(x) = \dfrac{3}{x}$ **B** $f^{-1}(x) = \dfrac{1}{x} + 3$ **C** $f^{-1}(x) = x - 3$ **D** $f^{-1}(x) = \dfrac{1}{x+3}$

3 If the roots of the quadratic equation $x^2 + 3x + k - 1 = 0$ are consecutive, evaluate k.

A $k = -1$ **B** $k = 1$ **C** $k = 2$ **D** $k = 3$

4 Write $p(x) = x^4 + 4x^3 - 14x^2 - 36x + 45$ as a product of its factors.

5 If α, β and γ are the roots of $x^3 - 3x^2 + x - 9 = 0$, find:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \alpha\gamma + \beta\gamma$ **d** $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

6 A monic polynomial $P(x)$ of degree 3 has zeros –2, 1 and 6. Write down the polynomial.

7 **a** Divide $P(x) = x^4 + x^3 - 19x^2 - 49x - 30$ by $x^2 - 2x - 15$.

b Hence, write $P(x)$ as a product of its factors.

8 Find the inverse function of $f(x) = 3 - 2x$.

9 For the polynomial $P(x) = x^3 + 2x^2 - 3x$, find:

a the degree

b the coefficient of x

c the zeros

d the leading term.

10 Sketch the graph of $f(x) = (x - 2)(x + 3)^2$, showing the intercepts.

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11 If $ax^4 + 3x^3 - 48x^2 + 60x = 0$ has a double root at $x = 2$, find:

a the value of a **b** the sum of the other 2 roots.

12 a State the domain and range of $y = \sqrt{x-1}$.

b Find the inverse of this function and state its domain and range.

13 a Find the domain and range of $y = \dfrac{1}{x+2}$.

b Find the inverse function.

c Find the domain and range of the inverse function.

14 Show that $x + 7$ is not a factor of $x^3 - 7x^2 + 5x - 4$.

15 If the sum of 2 roots of $x^4 + 2x^3 - 8x^2 - 18x - 9 = 0$ is 0, find the roots of the equation.

16 a Find the domain over which the curve $y = x^2 - 4x$ is monotonic increasing.

b Find the inverse function over this domain.

17 If $p(x) = x^3 - 1$ and $q(x) = 2x + 5$, evaluate:

a $p^{-1}(7)$ **b** $q^{-1}(p(3))$

18 The polynomial $f(x) = ax^2 + bx + c$ has zeros 4 and 5, and $f(-1) = 60$. Evaluate a, b and c.

19 Find the x- and y-intercepts of the curve $y = x^3 - 3x^2 - 10x + 24$.

20 Divide $p(x) = 3x^5 - 7x^3 + 8x^2 - 5$ by $x - 2$ and write $p(x)$ in the form $p(x) = (x - 2)\,a(x) + b(x)$.

21 When $8x^3 - 5kx + 9$ is divided by $x - 2$ the remainder is 3. Evaluate k.

22 Write $P(x) = x^5 + 2x^4 + x^3 - x^2 - 2x - 1$ as a product of its factors.

23 By restricting the domain of $f(x) = x^2 - 4$ to monotonic decreasing, find its inverse function.

24 Find the zeros of $g(x) = -x^2 + 9x - 20$.

25 Sketch the graph of $P(x) = 2x(x - 3)(x + 5)$, showing intercepts.

26 Find the value of k if the remainder is -4 when $x^3 + 2x^2 - 3x + k$ is divided by $x - 2$.

27 The sum of 2 roots of $x^4 - 7x^3 + 5x^2 - x + 3 = 0$ is 3. Find the sum of the other 2 roots.

28 The leading term of a polynomial is $3x^3$ and there is a double root at $x = 3$. Sketch a graph of the polynomial.

29 A polynomial $P(x)$ has a triple root at $x = -6$.

a Write an expression for $P(x)$.

b If $P(x)$ has leading coefficient 3 and degree 4, sketch a graph showing this information.

30 Draw an example of a polynomial with leading term $3x^5$.

31 If $f(x) = x^3$, show that $f[f^{-1}(x)] = f^{-1}[f(x)] = x$.

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6. CHALLENGE EXERCISE

1 **a** Write the polynomial $P(u) = u^3 - 4u^2 + 5u - 2$ as a product of its factors.

b Hence or otherwise, solve $(x - 1)^3 - 4(x - 1)^2 + 5(x - 1) - 2 = 0$.

2 **a** Write $f(u) = u^3 - 13u^2 + 39u - 27$ as a product of its factors.

b Hence or otherwise, solve $3^{3x} - 13(3^{2x}) + 39(3^x) - 27 = 0$.

3 Find the points of intersection between the polynomial $P(x) = x^3 + 5x^2 + 4x - 1$ and the line $3x + y + 4 = 0$.

4 Divide $6x^2 - 3x + 1$ by $3x - 2$.

5 If $P(x) = ax^3 + bx^2 + cx + d$ has a remainder of 8 when divided by $x - 1$, $P(2) = 17$, $P(-1) = -4$ and $P(0) = 5$, evaluate a, b, c and d.

6 If α, β and γ are roots of the cubic equation $2x^3 + 8x^2 - x + 6 = 0$, find:

a $\alpha\beta\gamma$

b $\alpha^2 + \beta^2 + \gamma^2$

7 Find the value of a if $(x + 1)(x - 2)$ is a factor of $2x^3 - x^2 + ax - 2$.

8 Prove that if $x - k$ is a factor of polynomial $P(x)$, then $P(k) = 0$.

9 Sketch a graph of a polynomial with a double root at $x = k_1$ and a double root at $x = k_2$, if the polynomial is monic, has even degree, and $k_2 > k_1$.

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