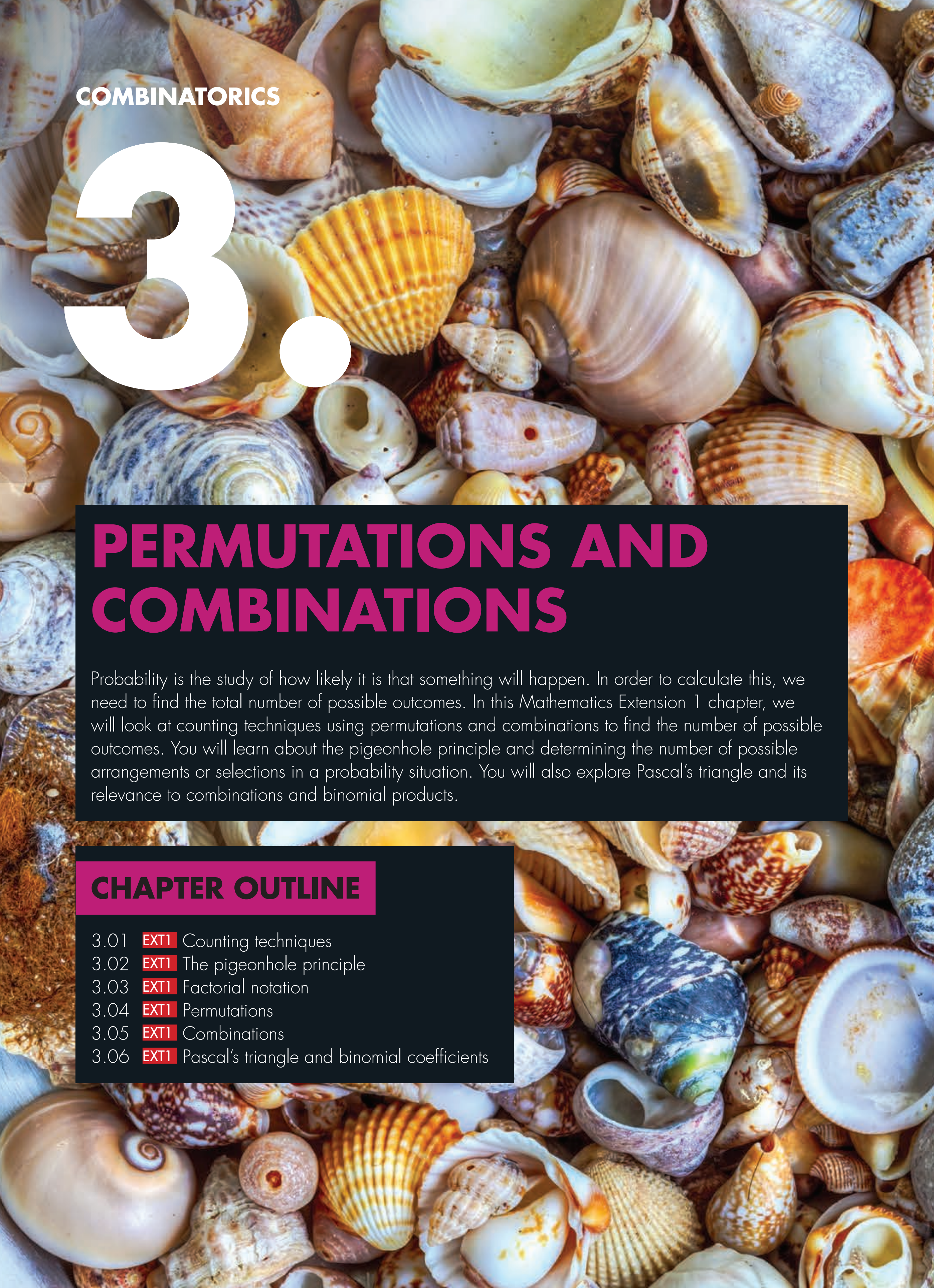

COMBINATORICS

3. PERMUTATIONS AND COMBINATIONS

Probability is the study of how likely it is that something will happen. In order to calculate this, we need to find the total number of possible outcomes. In this Mathematics Extension 1 chapter, we will look at counting techniques using permutations and combinations to find the number of possible outcomes. You will learn about the pigeonhole principle and determining the number of possible arrangements or selections in a probability situation. You will also explore Pascal's triangle and its relevance to combinations and binomial products.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- EXT1 use factorials and other counting techniques to find numbers of arrangements
- EXT1 use the pigeonhole principle to solve problems
- EXT1 distinguish between permutations and combinations and use them to find numbers of arrangements and selections to calculate simple probabilities
- EXT1 identify the relationship between Pascal's triangle and binomial coefficients

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EXT1 TERMINOLOGY

arrangements: Different ways of organising objects

binomial expansion: The algebraic expansion of powers of a binomial expression; for example, $(3x - 5)^7$

combinations: Arrangements of objects when order is not important

factorial: The product of n consecutive positive integers from n down to 1. For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

fundamental counting principle: If one event can occur in p ways and a second independent event can occur in q ways, then the two successive events can occur in $p \times q$ different ways

ordered selections: Selections that are taken in a particular position or order

permutations: Arrangements of objects when order is important

unordered selections: Selections that are made when the order of arrangements is not important or relevant

EXT1 3.01 Counting techniques

To find the probability of an event happening, we compare the number of ways the event can occur with the total number of possible outcomes (the sample space):

$$\text{Probability of an event} = \frac{\text{Number of ways the event can occur}}{\text{Total number of possible outcomes}}$$

The hardest part of calculating probabilities is finding all the possible outcomes. This can become quite difficult when the numbers of outcomes are large. There are some counting techniques that help in these cases.

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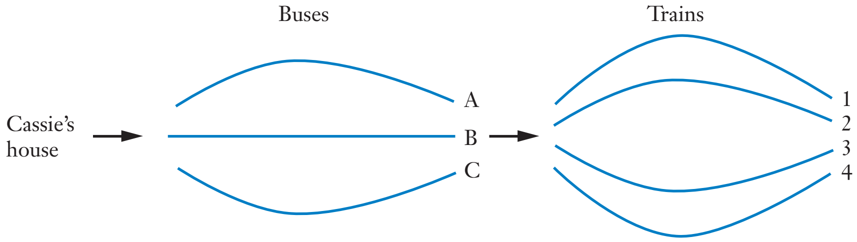
INVESTIGATION

COUNTING

1 Cassie needs to catch a bus and a train to work. There are 3 different buses she could catch into town. When she arrives in town, she needs to catch one of 4 trains to work.

If there are 3 buses and 4 trains possible for Cassie to catch, in how many ways is it possible for her to travel to work?



2 A restaurant offers 3 entrees, 4 main meals and 2 desserts. Every time Rick eats at the restaurant he chooses to eat a different combination of courses. How many times would he need to go to the restaurant to cover all possible combinations?

The **fundamental counting principle** comes from the product rule of probability, which you will study in detail in Chapter 9, *Probability*. The investigation above shows how it works.

For example, Cassie could travel by 3×4 or 12 different routes:

A1	A2	A3	A4
B1	B2	B3	B4
C1	C2	C3	C4

If one event can happen in p different ways and another event can happen in q different ways, then the 2 successive events can happen in pq different ways.

We can generalise even further to many events:

Fundamental counting principle

If one event can happen in a different ways, a second event in b different ways, a third event in c different ways and so on, then the successive events can happen in abc … different ways.

EXAMPLE 1

a The number plate on a car has 2 letters, followed by 4 numbers. How many different number plates of this type are possible?

b I have 12 pairs of earrings, 3 necklaces, 8 rings and 2 watches in my jewellery box.

- **i** If I can wear any combination of earrings, necklaces, rings and watches, how many different sets of jewellery can I wear?
- **ii** If my friend makes a guess at the combination of jewellery that I will wear, what is the probability that she will guess correctly?

c A restaurant serves 5 different types of entree, 12 main courses and 6 desserts.

- **i** If I order any combination of entree, main course and dessert at random, how many different combinations are possible?
- **ii** If my friend makes 3 guesses at which combination I will order, what is the probability that she will guess correctly?

Solution

a There are 26 letters and 10 numbers (0 to 9) possible for each position in the number plate. Using the fundamental counting principle:

$$\begin{aligned}\text{Total number} &= 26 \times 26 \times 10 \times 10 \times 10 \times 10\\ &= 26^2 \times 10^4\\ &= 6\,760\,000\end{aligned}$$

So 6 760 000 number plates are possible.

b **i** $\begin{aligned}\text{Total number} &= 12 \times 3 \times 8 \times 2\\ &= 576\end{aligned}$

ii The friend makes 1 guess and there are 576 possible outcomes.

$$P(\text{correct guess}) = \frac{1}{576}$$

c **i** $\begin{aligned}\text{Total number of combinations} &= 5 \times 12 \times 6\\ &= 360\end{aligned}$

ii The friend makes 3 guesses and there are 360 possible outcomes.

$$\begin{aligned}P(\text{correct guess}) &= \frac{3}{360}\\ &= \frac{1}{120}\end{aligned}$$

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Sometimes an outcome depends on what happens previously.

EXAMPLE 2

a To win a trifecta bet in a race, a person has to pick the horses that come first, second and third in the race, in the correct order. If a race has 9 horses, how many different trifecta bets are possible?

b A group of 15 people attend a concert and 3 of them are randomly chosen to receive a free backstage pass. The first person receives a gold pass, the second one a silver pass and the third one a bronze pass. In how many different ways can the passes be given out?

c In Lotto Strike, a machine contains 45 balls, each with a different number from 1 to 45. Players must guess the first 4 numbers to be drawn, in the correct order, to win first prize.

- **i** In how many ways can 4 balls be randomly drawn in order?
- **ii** Lisa has 3 entries in the same draw of Lotto Strike. What is the probability that she will win first prize?

Solution

a Any of the 9 horses could come first.

Any of the remaining 8 could come second.

Any of the remaining 7 horses could come third.

$$\begin{aligned}\text{Total ways} &= 9 \times 8 \times 7 \\ &= 504\end{aligned}$$

b Any of the 15 people can receive the first pass.

There are 14 people left who could receive the second pass.

Similarly there are 13 people who could receive the third pass.

$$\begin{aligned}\text{Total number of possibilities} &= 15 \times 14 \times 13 \\ &= 2730\end{aligned}$$

c **i** The first ball could be any of the 45 balls.

The second could be any of the remaining 44 balls and so on.

$$\begin{aligned}\text{The number of ways} &= 45 \times 44 \times 43 \times 42 \\ &= 3\ 575\ 880\end{aligned}$$

ii $$\begin{aligned}P(\text{first prize}) &= \frac{3}{3\,575\,880} \\ &= \frac{1}{1\ 191\ 960}\end{aligned}$$

EXT1 Exercise 3.01 Counting techniques

1 A password has 4 letters. How many passwords are possible?

2 A motorcycle number plate is made up of 2 letters followed by 2 numbers. How many number plates of this type are available?

3 A password can have up to 5 letters followed by 4 numbers. If I could use any letter of the alphabet or number, how many different passwords could be formed? Leave your answer in index form.

4 A witness saw most of the number plate on a getaway car except for the first letter and the last number. How many different cars do the police need to check in order to find this car?

5 A certain brand of computer has a serial number made up of 10 letters then 15 numbers. How many computers with this type of serial number can be made? Leave your answer in index form.

6 Victoria has postcodes starting with 3. How many different postcodes are available in Victoria?

7 A country town has telephone numbers starting with 63 followed by any 6 other numbers from 0 to 9. How many telephone numbers are possible in this town?

8 Jarred has 12 tops, 5 pairs of jeans and 5 pairs of shoes in his wardrobe. If he chooses a top, pair of jeans and shoes at random, how many combinations are possible?

9 A car manufacturer produces cars in 8 different colours, with either manual or automatic gear transmission, and 4 different types of wheels. How many different combinations can it produce?

10 A PIN has 4 numbers. If I forget my PIN I am allowed 3 tries to get it right. Find the probability that I get it within the 3 tries.

11 A restaurant offers 7 main courses and 4 desserts, as well as 3 different types of coffee.

- **a** How many different combinations of main course, dessert and coffee are possible?
- **b** Find the probability that I randomly pick the combination most often voted favourite.

12 A telephone number in a capital city can start with a 9 and has 8 digits altogether.

- **a** How many telephone numbers are possible?
- **b** If I forget the last 3 digits of my friend's telephone number, how many numbers would I have to try for the correct number?

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13 A company manufactures 20 000 000 computer chips. If it uses a serial number on each one consisting of 10 letters, will there be enough serial numbers for all these chips?

14 A password consists of 2 letters followed by 5 numbers. What is the probability that Izak randomly guesses the correct password?

15 A city has a population of 3 500 000. How many digits should its telephone numbers have so that every person can have one?

16 A manufacturer of computer parts puts a serial number on each part, consisting of 3 letters, 4 numbers then 4 letters. The number of parts sold is estimated as 5 million. Will there be enough combinations on this serial number to cope with these sales?

17 A bridal shop carries 12 different types of bridal dresses, 18 types of veils and 24 different types of shoes. If Kate chooses a combination of dress, veil and shoes at random, what is the probability that she chooses the same combination as her friend Yasmin?

18 Kate chooses a different coloured dress for each of her 3 bridesmaids. If the colours are randomly given to each bridesmaid, how many different possibilities are there?

19 In a computer car race game, the cars that come first, second and third are awarded at random. If there are 20 cars, how many possible combinations of first, second and third are there?

20 Jordan only has 4 different chocolates left and decides to randomly choose which of his 6 friends will receive one each. How many possible ways are there in which he can give the chocolates away?

21 Three different prizes are given away at a concert by taping them underneath random seats. If there are 200 people in the audience, in how many ways can these prizes be won?

22 There are 7 clients at a barber shop. If there are 3 barbers working, in how many ways could 3 clients be selected to have their haircut first?

23 A family of 5 people each choose a flavour of ice cream from vanilla, strawberry and chocolate. In how many ways can this happen?

24 A set of cards is numbered 1 to 100 and 2 chosen at random.

- **a** How many different arrangements of ordered pairs are possible?
- **b** What is the probability that a particular ordered pair is chosen?

25 Each of 10 cards has a letter written on it from A to J. If 3 cards are selected in order at random, find the probability that they spell out CAB.

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EXT1 3.02 The pigeonhole principle

The **pigeonhole principle** is another useful counting technique.

Pigeonhole principle

If $n + 1$ or more pigeons are placed into n pigeonholes, then at least one pigeonhole must contain 2 or more pigeons.

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Proof

Suppose there are n pigeonholes and only one pigeon in each hole.

Then the maximum number of pigeons is n.

But there are $n + 1$ pigeons, so the assumption that there is only one pigeon per hole is wrong.

Therefore at least one pigeonhole must contain 2 or more pigeons.

EXAMPLE 3

a A bag contains green, black, yellow, white, red and blue jellybeans. How many jellybeans must Keira take out of the bag so that she is sure to take 2 of the same colour?

b Is it certain that at least 2 students will have the same birthday in a school with 750 students?

c A paragraph contains 33 words. Show that there must be at least 2 words that start with the same letter.

d A square with side length 2 cm has 9 points drawn at random inside the square. Show that it is possible for 3 of these points to form a triangle with an area less than 1 cm^2.

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Solution

a There are 6 different colours. Using the pigeonhole principle, Keira must take out 7 jellybeans to be sure of taking 2 of the same colour (since the first 6 could be different colours).

b There are 366 possible birthdays, including leap years.

So there only need to be 367 people for at least 2 to have the same birthday.

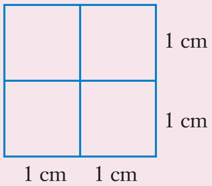
Since there are 750 students, which is more than 366, at least 2 students must share the same birthday.

c There are 26 letters in the alphabet.

So there only need to be 27 words so that at least 2 will start with the same letter.

Since the paragraph contains 33 words, which is more than 26, at least 2 must start with the same letter.

d Divide the square into 4 smaller squares with area 1 cm^2 as shown.



For 3 points to form a triangle with area less than 1 cm^2, they must lie within the same smaller square (since the triangle inside will have a smaller area than the square's area).

When placing 4 points inside the square, it is possible that each could lie in a different smaller square. Placing the next 4 points could also result in each being in a different smaller square. This means that now the smaller squares must have at least 2 points inside.

The next (9th) point must go into one of the 4 smaller squares, so even if there were only 2 points in each smaller square previously, now there must be 3 points in at least one of the smaller squares.

So it is possible to form a triangle from these 3 points (out of the 9 points) with an area less than 1 cm^2.

Generalised pigeonhole principle

If n pigeons are placed into k pigeonholes, where $n > k$, then at least one pigeonhole must contain at least $\frac{n}{k}$ pigeons.

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Proof

Suppose there is only one pigeon in each of the k pigeonholes.

Then the maximum number of pigeons is k.

But $n > k$, so the assumption of one pigeon per hole is wrong.

Therefore, there must be more than one pigeon in at least one hole.

The average number of pigeons per pigeonhole must be $\dfrac{\text{Number of pigeons}}{\text{Number of pigeonholes}} = \dfrac{n}{k}$.

Each pigeonhole will be below or above the average, or on the average.

So at least one pigeonhole must contain at least $\dfrac{n}{k}$ pigeons.

(If $\dfrac{n}{k}$ is not a whole number, then because some pigeonholes will be above the average we can round *up* to the next whole number.)

EXAMPLE 4

a A group of 75 people in a singing contest are placed into different audition rooms according to their category:

- males
- females
- children
- groups

If there are at least x people in one of the rooms, find the value of x.

b A group of 117 people rated a TV show from 1 to 5. Find r if there were at least r people who gave the same rating.

Solution

a There are 75 people and 4 rooms.

So $n = 75$ and $k = 4$.

$\dfrac{n}{k} = \dfrac{75}{4} = 18\dfrac{3}{4}$

$x \geq 18\dfrac{3}{4} = 19$ (as x is a whole number).

If $\dfrac{n}{k}$ is not a whole number, always round *up*.

There are at least 19 people in at least one of the rooms.

b $n = 117$ and $k = 5$

$\dfrac{n}{k} = \dfrac{117}{5} = 23\dfrac{2}{5}$

$r \geq 23\dfrac{2}{5} = 24$ (as r is a whole number).

There were at least 24 people who gave the same rating.

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DID YOU KNOW?

The Dirichlet principle

The pigeonhole principle is also called the **Dirichlet principle**. The German mathematician Johann Dirichlet (1805–1859) was the first person to come up with this principle, in 1834. He was also involved in other branches of mathematics.

Research his other contributions and his place in the mathematics of the times.

EXT1 Exercise 3.02 The pigeonhole principle

1 A set of blocks contains red, blue, yellow and green blocks. How many blocks must Stevie choose at random to ensure that there are at least 2 blocks with the same colour?

2 A national committee is made up of members from NSW, Victoria, Queensland, South Australia and Tasmania. How many committee members are needed so that at least 2 of them must be from the same state?

3 A school has 9 different sports for its weekly sports afternoon. How many students would you need to survey to ensure that at least 2 of them are from the same sporting group?

4 A farm has 20 sheep, 20 cows and 20 pigs in a paddock. Show that if 4 animals escape from the paddock, at least 2 must be the same type of animal.

5 Show that if a wardrobe contains 8 pairs of black socks and 8 pairs of white socks, only 3 need to be chosen to find a pair of socks with the same colour.

6 Show that if you choose 5 cards from a deck of playing cards, then at least 2 must be the same suit.

7 Show that if eye colour can be described as blue, green, hazel or brown, then only 5 people need to be chosen for at least 2 to have the same eye colour.

8 A farmer picks 83 oranges and places them in barrels according to their size: small, medium, large and extra large. Find the value of x if at least one barrel has at least x oranges.

9 The long-term car park at the airport has 1024 cars, in sections labeled A, B, C, … M. Find the minimum number of cars parked in at least one of the sections.

10 A herd of 129 dairy cows are put into 3 pens: those too young to milk, those ready to be milked and those already milked. Find the minimum number of cows in at least one of the pens.

11 On New Year's Eve there were 9 different parks that were best for watching the fireworks. If there were 2495 people in these parks, find the minimum number of people in at least one of these parks.

12 The numbers 1 to 30 are divided by 7 and the remainder recorded. Find the value of x if at least x of the numbers have the same remainder.

13 There are n people placed in 8 levels of karate. If there are at least 29 people in at least one level, find the value of n.

14 In a survey of 450 people, there were at least 35 who preferred the same type of take-away food. How many different types of take-away foods were surveyed?

15 A group of friends split up into different groups, with some going to the cinema, some going to a concert and others going out to dinner. How many friends must you select so that at least 3 of them went to the same event?

WS Factorial notation

EXT1 3.03 Factorial notation

Counting outcomes when repetition or replacement is allowed is straightforward, even when the numbers become very large.

EXAMPLE 5

A card is drawn at random from a set of 25 cards numbered 1 to 25. The card is then replaced before the next is selected. How many possible outcomes are there if 25 cards are chosen this way? Answer in scientific notation, correct to 3 significant figures.

Solution

Each time there is a card drawn, there are 25 possibilities.

$$\begin{aligned}\text{Total number} &= 25 \times 25 \times 25 \times \ldots \times 25 \text{ (25 times)}\\ &= 25^{25}\\ &\approx 8.88 \times 10^{34}\end{aligned}$$

When there is no repetition or replacement, the calculations can be long.

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EXAMPLE 6

A card is drawn at random from a set of 25 cards numbered 1 to 25. The card is not replaced before the next is drawn. How many possible outcomes are there if all 25 cards are drawn? Answer in scientific notation, correct to 3 significant figures.

Solution

For the first card, there are 25 possibilities.

For the second card, there are only 24 possibilities because one card has already been drawn.

For the third card, there are 23 possibilities, and so on.

$$\begin{aligned}\text{Total number of possible outcomes} &= 25 \times 24 \times 23 \times \ldots\ 3 \times 2 \times 1 \\ &\approx 1.55 \times 10^{25}\end{aligned}$$

The product of consecutive whole numbers $25 \times 24 \times 23 \times \ldots \times 3 \times 2 \times 1$ is called '25 factorial' and is written as '25!'.
Factorial notation allows us to write the number of possible outcomes when selecting all objects in order with no replacement or repetition.

Factorial notation

The number of ways of selecting n objects in order with no replacement or repetition is $n!$ (n factorial).

$$n! = n(n-1)(n-2)(n-3)(n-4)\ \ldots\ 3 \times 2 \times 1$$

Mathematicians find it convenient to define zero factorial as being equal to 1.

$$0! = 1$$

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EXAMPLE 7

a Evaluate:

i 4!   **ii** 7!

iii 25! (in scientific notation correct to 3 significant figures.)

b A group of 9 teenagers is waiting to be served in a café. They are each randomly assigned a number from 1 to 9.

i In how many ways is it possible for the numbers to be assigned?

ii One of the group needs to be served quickly because he has to leave. If he is given the first number, in how many ways is it possible for the numbers to be assigned?

Solution

a **i** $4! = 4 \times 3 \times 2 \times 1$

$= 24$

ii $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5040$

iii

Operation	Casio scientific	Sharp scientific
Enter data	25 [x!] [=]	25 [2ndF] [x!] [=]

$25! \approx 1.55 \times 10^{25}$

b **i** The first number could be assigned 9 ways.

The second number could be assigned 8 ways and so on.

Total ways $= 9!$

Operation	Casio scientific	Sharp scientific
Enter data:	9 [x!] [=]	9 [2ndF] [x!] [=]

$9! = 362\ 880$

So there are 362 880 ways for the numbers to be assigned.

ii One of the group is given the first ticket. This can only happen in one way.

The second number could be assigned 8 ways, and so on.

Total ways $= 1 \times 8!$

$= 40\ 320$

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EXT1 Exercise 3.03 Factorial notation

1 Evaluate:

a $6!$ **b** $10!$ **c** $0!$ **d** $8! - 7!$ **e** $5 \times 4!$

f $\frac{7!}{4!}$ **g** $\frac{12!}{5!}$ **h** $\frac{13!}{4!9!}$ **i** $\frac{8!}{3!5!}$ **j** $\frac{11!}{4!7!}$

2 A group of 9 jockeys are each given a set of riding colours to wear. If these are given out randomly, how many different arrangements are possible?

3 Each of 6 people at a restaurant is given a different-coloured glass. How many possible combinations are there?

4 A mountain trail has room for only one person at a time. If 12 people are waiting at the bottom of the trail and are picked at random to start out, in how many ways can this happen?

5 A dog walker has 5 dogs and 5 leashes. In how many different ways is it possible to put a leash on each dog?

6 There are 11 actors in a play and each receives a script highlighting different parts.

a In how many different ways could the scripts be handed out?

b Anthony, the director, also needs a script. In how many ways could the scripts be handed out for the actors and the director?

7 A row of seats in a theatre seats 8 people. In how many ways could a group of 8 friends be seated at random in this row?

8 A group of 7 people line up to do karaoke. If they are each given a song at random to sing, how many possible outcomes are there?

9 A kindergarten class has a rabbit, a mouse and a parrot. Three children are selected to take these pets home for the holidays. If the pets are given out at random to these children, how many different ways are possible?

10 A group of 6 students are each given a different topic for a speech. In how many ways can the 6 topics be given to the 6 students?

11 In a chorus for a school musical, 7 students each wear a different mask. In how many different ways can the masks be worn by these students?

12 If 15 people play a game of Kelly pool, each person in turn chooses a number at random between 1 and 15. In how many different ways can this occur? Answer in scientific notation, correct to one decimal place.

13 a A school talent quest has 11 performers and each one is randomly given the order in which to perform. In how many ways can the order of performances be arranged?

b If one performer is chosen to perform first, in how many ways can the others be arranged?

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14 A group of 6 friends sit in the same row at a concert.

a In how many different ways can they arrange themselves?

b If one friend must sit on the centre aisle, in how many ways can they be arranged?

15 A group of 8 friends go to a restaurant and sit at a round table. If the first person can sit anywhere, in how many ways can the others be arranged around the table?

16 In a pack of cards, the 4 aces are taken out and shuffled.

a What is the probability of picking out the ace of hearts at random?

b If all the aces are arranged in order, what is the probability of guessing the correct order?

Photo courtesy Margaret Grove

17 At a wedding, each of the 12 tables is to have a centrepiece with a different coloured rose.

a In how many different ways can the roses be arranged at random?

b What is the probability that the bride will have a pink rose at her table?

18 In a maths exam, a student has to arrange 5 decimals in the correct order. If he has no idea how to do this and arranges them randomly, what is the probability that he makes the right guess for all the decimals?

19 In a car race, the fastest car is given pole position and the other cars are given their starting positions at random. If there are 14 cars altogether, in how many ways can this be arranged?

20 Show that:

a $\frac{8!}{4!} = 8 \times 7 \times 6 \times 5$

b $\frac{11!}{6!} = 11 \times 10 \times 9 \times 8 \times 7$

c $\frac{n!}{r!} = n(n-1)(n-2)(n-3) \ldots (r+1)$ where $n > r$

d $\frac{n!}{(n-r)!} = n(n-1)(n-2)(n-3) \ldots (n-r+1)$ where $n > r$

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EXT1 3.04 Permutations

Factorial notation is useful for writing the number of possible outcomes when arranging **all** objects in order without replacement. It becomes slightly more complex when we arrange only **some** of the objects in order without replacement.

WS Counting techniques

WS Permutation calculations

WS Permutations

EXAMPLE 8

13 cards are chosen at random from 20 cards without replacement. Find the possible number of ways the cards can be chosen.

Solution

The first card can be any of the 20 numbers.

The second card can be any of the remaining 19 numbers.

The third can be any of the remaining 18 numbers, and so on.

Then the number of ways the cards can be chosen $= 20 \times 19 \times 18 \times 17 \times \ldots \times 8$

$\approx 4.8 \times 10^{14}$

A **permutation** describes an **arrangement** or **ordered selection** of r objects from a total of n objects without replacement or repetition.

Permutations

The permutation ^nP_r is the number of ways of making ordered selections of r objects from a total of n objects.

$$\begin{aligned} {}^nP_r &= n \times (n-1) \times (n-2) \times (n-3) \times \ldots \qquad (r \text{ times}) \\ &= n(n-1)(n-2)(n-3) \times \ldots \times (n-r+1) \end{aligned}$$

$${}^nP_r = \frac{n!}{(n-r)!}$$

Proof

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)(n-3) \times \ldots \times (n-r+1) \\ &= n(n-1)(n-2)(n-3) \times \ldots \times (n-r+1) \times \frac{(n-r)(n-r-1)(n-r-2) \times \ldots \times 3 \times 2 \times 1}{(n-r)(n-r-1)(n-r-2) \times \ldots \times 3 \times 2 \times 1} \\ &= \frac{n(n-1)(n-2)(n-3) \times \ldots \times (n-r+1)(n-r)(n-r-1)(n-r-2) \times \ldots \times 3 \times 2 \times 1}{(n-r)(n-r-1)(n-r-2) \times \ldots \times 3 \times 2 \times 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

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A special case of this result is:

$$^nP_n = n!$$

Proof

$$\begin{aligned} ^nP_r &= \frac{n!}{(n-r)!} \\ \therefore\ ^nP_n &= \frac{n!}{(n-n)!} \\ &= \frac{n!}{0!} \\ &= \frac{n!}{1} \\ &= n! \end{aligned}$$

EXAMPLE 9

a Evaluate $^9P_4$.

b **i** Find the number of arrangements of 3 digits that can be formed using the digits 0 to 9 if each digit can be used only once.

ii How many 3-digit numbers greater than 700 can be formed?

Solution

a
$$\begin{aligned} ^9P_4 &= \frac{9!}{(9-4)!} \\ &= \frac{9!}{5!} \\ &= \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1} \\ &= 9\times8\times7\times6 \\ &= 3024 \end{aligned}$$

Operation	Casio scientific	Sharp scientific
Enter data	9 SHIFT ^nP_r 4 =	9 2ndF ^nP_r 4 =

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b **i** There are 10 digits from 0 to 9.

The first digit can be any of the 10 digits.

The second digit can be any of the remaining 9 digits.

The third digit can be any of the remaining 8 digits.

$$\begin{aligned}\text{Total permutations} &= 10 \times 9 \times 8 \\ &= 720\end{aligned}$$

or

$$\begin{aligned}{}^{10}P_3 &= \frac{10!}{(10-3)!} \\ &= \frac{10!}{7!} \\ &= 720\end{aligned}$$

ii The first digit must be 7 or 8 or 9 (3 possible digits).

The second digit can be any of the remaining 9 digits.

The third digit can be any of the remaining 8 digits.

$$\begin{aligned}\text{Total arrangements} &= 3 \times 9 \times 8 \\ &= 216\end{aligned}$$

Using permutations:

There are 3 ways to get the first digit.

The possible arrangements of the remaining 2 digits is ${}^9P_2$.

$$\begin{aligned}\text{Total arrangements} &= 3 \times {}^9P_2 \\ &= 3 \times 72 \\ &= 216\end{aligned}$$

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WS Circular permutations

WS Permutations with repetitions

Permutations with restrictions

Some examples need very careful counting. As you saw in the above example, sometimes we can use permutations and sometimes factorials.

EXAMPLE 10

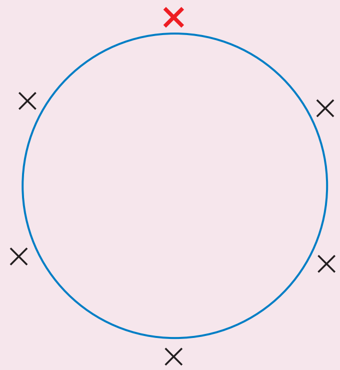
a **i** In how many ways can 6 people sit around a circular table?

ii If seating is random, find the probability that 3 particular people will sit together.

b In how many ways can the letters of the word EXCEPTIONAL be arranged?

Solution

a **i** The first person can sit anywhere around the table so we only need to arrange the other 5 people.



The second person can sit in any of the 5 remaining seats.

The third person can sit in any of the remaining 4 seats and so on.

$$\begin{aligned}\text{Total arrangements} &= 5! \\ &= 120\end{aligned}$$

ii The 3 people can sit anywhere around the table together in $3 \times 2 \times 1$ or $3!$ ways.

The remaining 3 people can sit together in $3!$ ways.

$$\begin{aligned}\text{Total arrangements} &= 3! \times 3! \\ &= 36\end{aligned}$$

$$\begin{aligned}P(\text{3 sit together}) &= \frac{36}{120} \\ &= \frac{3}{10}\end{aligned}$$

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b EXCEPTIONAL has 11 letters with the letter E repeated.

If each E was different, i.e. E_1 and E_2, then there would be 11! arrangements.

However, we cannot tell the difference between the 2 Es.

Since there are 2! ways of arranging the Es, then there are 2! arrangements of the word EXCEPTIONAL that are identical. We need to divide by 2! to eliminate these identical arrangements.

$$\text{Total arrangements} = \frac{11!}{2!} = 19\,958\,400$$

Permutations involving repeated objects

The number of different ways of arranging n objects in which a of the objects are of one kind, b objects are of another kind, c of another kind and so on, is given by $\frac{n!}{a!b!c!\ldots}$ where $a + b + c + \ldots \le n$.

EXAMPLE 11

Find the number of ways that the letters of the place name ULLADULLA can be arranged.

Solution

There are 9 letters, including 4 Ls, 2 As and 2 Us. There are 9! ways of arranging the letters, with 4! ways of arranging the Ls, 2! of arranging the As and 2! ways of arranging the Us.

$$\text{Total arrangements} = \frac{9!}{4!2!2!} = 3780$$

There are different ways of working out the number of arrangements. Sometimes it is just a matter of drawing a diagram or counting carefully.

Permutations

EXAMPLE 12

A bag contains 5 balls of different colours – red, yellow, blue, green and white. In how many ways can these 5 balls be arranged:

a with no restrictions

b if the yellow ball must be first

c if the first ball must not be red or white

d if blue and green must be together

e if red, blue and green must be together?

Solution

a The first can be any of the 5 balls.
The second can be any of the remaining 4 balls and so on.

$$\begin{aligned}\text{Total arrangements} &= 5! \\ &= 120\end{aligned}$$

b The first ball must be yellow, so there is only 1 way of arranging this.
The second ball can be any of the remaining 4 balls.
The third ball can be any of the remaining 3 balls and so on.

$$\begin{aligned}\text{Total arrangements} &= 4! \\ &= 24\end{aligned}$$

c The first ball could be yellow, blue or green so there are 3 possible arrangements.
The second ball could be any of the remaining 4 balls and so on.

$$\begin{aligned}\text{Total arrangements} &= 3 \times 4! \\ &= 72\end{aligned}$$

d When two objects must be together, we treat them as a single object with 2! possible arrangements.
So we arrange 4 balls in 4! ways: R, Y, BG and W.
But there are 2! ways in which to arrange the blue and green balls.

$$\begin{aligned}\text{Total arrangements} &= 4! \times 2! \\ &= 48\end{aligned}$$

e When three objects are together, we treat them as a single object with 3! possible arrangements.
We are then arranging 3 balls in 3! ways: RBG, Y, W.
But there are 3! ways in which to arrange the red, blue and green balls.

$$\begin{aligned}\text{Total arrangements} &= 3! \times 3! \\ &= 36\end{aligned}$$

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EXT1 Exercise 3.04 Permutations

1 Evaluate each permutation.

a ${}^6P_3$ **b** ${}^5P_2$ **c** ${}^8P_3$ **d** ${}^{10}P_7$ **e** ${}^9P_6$

f ${}^7P_5$ **g** ${}^8P_6$ **h** ${}^{11}P_8$ **i** ${}^9P_1$ **j** ${}^6P_6$

2 A set of 26 cards, each with a different letter of the alphabet, is placed into a hat and cards drawn out at random without replacement. Find the number of 'words' possible if selecting:

a 2 cards **b** 3 cards

c 4 cards **d** 5 cards.

3 A random 3-digit number is made from cards containing the numbers 0 to 9.

a In how many ways can this be done if the cards cannot be used more than once and 0 cannot be the first number?

b How many numbers over 400 can be made?

c How many numbers less than 300 can be made?

4 A set of 5 cards, each with a number from 1 to 5 on it, is placed in a box and 2 drawn out at random. Find the possible number of combinations:

a altogether **b** of numbers greater than 50

c of odd numbers **d** of even numbers.

5 a How many arrangements of the letters A, B, C and D are possible if no letter can be used twice?

b How many arrangements of any 3 of these letters are possible?

6 A 4 digit number is to be selected at random from the numbers 0 to 9 with a non-zero first digit and no repetition.

a How many arrangements can there be?

b How many arrangements of numbers over 6000 are there?

c How many arrangements of numbers less than 8000 are there?

7 The numbers 1, 2, 3, 4 and 5 are arranged in a line. How many arrangements are possible if:

a there is no restriction **b** the number is less than 30 000

c the number is greater than 20 000 **d** the number is odd

e any 3 numbers are selected at random?

8 There are 12 swimmers in a race.

a In how many ways could they finish?

b In how many ways could they come in first, second and third?

9 How many different ordered arrangements can be made from the word COMPUTER with:

a 2 letters? **b** 3 letters? **c** 4 letters?

10 How many different ordered arrangements can be made from these words?

a CENTIPEDE **b** ALGEBRA **c** TELEVISION
d ANTARCTICA **e** DONOR **f** BASKETBALL
g GREEDY **h** DUTIFUL **i** MANUFACTURER
j AEROPLANE

11 A group of friends queue in a straight line outside a night club. Find how many ways the friends can be arranged if there are:

a 4 friends **b** 7 friends **c** 8 friends
d 10 friends **e** 11 friends.

12 A group of friends go into a restaurant and are seated around a circular table. Find how many arrangements are possible if there are:

a 4 friends **b** 7 friends **c** 8 friends
d 10 friends **e** 11 friends.

13 A string necklace contains a circle of beads, but each possible arrangement of beads can also be worn back-to-front (flipped). Find the number of different arrangements possible with:

a 10 beads **b** 12 beads **c** 9 beads
d 11 beads **e** 13 beads.

14 In how many ways can a group of 6 people be arranged:

a in a line? **b** in a circle?

15 Find how many different ways a group of 9 people can be arranged in:

a a line **b** a circle.

16 In how many ways can a set of 10 beads be arranged:

a in a line?
b in a circle around the edge of a poster?
c on a bracelet?

17 **a** How many different arrangements can be made from the jack, queen, king and ace of hearts?

b If I choose 2 of these cards at random, how many different arrangements could I make?

c If I choose 3 of these cards at random, how many different arrangements could I make?

18 A group of 7 people sit around a table. In how many ways can they be arranged:

a with no restrictions? **b** if 2 people want to sit together?
c if 2 people cannot sit together? **d** if 3 people sit together?

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19 A group of 5 boys and 5 girls line up outside a cinema. In how many ways can they be arranged:

a with no restriction?

b if a particular girl stands in line first?

c if boys and girls alternate (with either a girl or boy in first place)?

20 Find the probability that if 10 people sit around a table at random, 2 particular people will be seated together.

21 A bookshelf is to hold 5 mathematics books, 8 novels and 7 cookbooks.

a In how many different ways could they be arranged? (Leave your answer in factorial notation.)

b If the books are grouped in categories, in how many ways can they be arranged? (Answer in factorial notation.)

c If one book is chosen at random, find the probability that it is a cookbook.

22 **a** How many different arrangements can be made from the numbers 3, 4, 4, 5 and 6?

b How many arrangements form numbers greater than 40 000?

c How many form numbers less than 50 000?

d If an arrangement is made at random, find the probability that it is less than 40 000.

23 Find the probability that an arrangement of the word LAPTOP will start with T.

24 What is the probability that, if a 3-letter 'word' is formed from the letters of PHYSICAL at random, it will be CAL?

25 A minibus has 6 forward-facing and 2 backward-facing seats. If 8 people use the bus, in how many ways can they be seated:

a with no restrictions

b if one person must sit in a forward-facing seat

c if 2 people must sit in a forward-facing seat?

26 If 3 letters of the word VALUED are selected at random, find the number of possible arrangements if:

a the first letter is D

b the first letter is a vowel.

27 The letters of the word THEORY are arranged randomly. Find the number of arrangements:

a with no restrictions

b if the E is at the beginning

c if the first letter is a consonant and the last letter is a vowel.

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28 Find the number of arrangements possible if x people are:

- **a** in a straight line
- **b** in a circle
- **c** in a circle with 2 people together
- **d** in a straight line with 3 people together
- **e** in a circle with 2 people not together.

29 **a** Use factorial notation to show that $\dfrac{{}^8P_3}{3!} = \dfrac{{}^8P_5}{5!}$.

b Prove that $\dfrac{{}^nP_r}{r!} = \dfrac{{}^nP_{n-r}}{(n-r)!}$.

30 Prove that ${}^{n+1}P_r = {}^nP_r + r\,{}^nP_{r-1}$.

WS Combination calculations

WS Combinations

WS Permutations and combinations

EXT1 3.05 Combinations

The permutation ${}^nP_r$ is the number of arrangements possible for an ordered selection of r objects from a total of n objects.

When the order is not important, for example when AB is the same as BA, the number of arrangements is called a **combination**.

A combination describes an **unordered selection** of r objects from a total of n objects without replacement or repetition.

EXAMPLE 13

a A committee of 2 is chosen from Scott, Rachel and Frankie. In how many ways can this be done?

b There are 3 vacancies on a school council and 8 people who are available. If the vacancies are filled at random, in how many ways can this happen?

Solution

a Number of ordered arrangements $= {}^3P_2$

$$= 6$$

However, a committee of Scott and Rachel is the same as a committee of Rachel and Scott. This is the same for all other arrangements of the committee. There are 2! ways of arranging each committee of 2 people.

To get the number of unordered arrangements, we divide the number of ordered arrangements by 2!

$$\text{Total arrangements} = \frac{{}^3P_2}{2!}$$

$$= 3$$

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b Number of ordered arrangements = ${}^8P_3$

However, order is not necessary here, since the 3 vacancies filled by, say, Henry, Amie and Wade, would be the same in any order.

There are 3! different ways of arranging Henry, Amie and Wade.

$$\text{So, total arrangements} = \frac{{}^8P_3}{3!} = 56$$

Combinations

The combination ${}^nC_r$ is the number of ways of making unordered selections of r objects from a total of n objects.

$$\begin{aligned} {}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n!}{(n-r)!r!} \end{aligned}$$

Proof

${}^nP_r$ is the ordered selection of r objects from n objects.

There are $r!$ ways of arranging r objects.

If order is unimportant, the unordered selection of r objects from n is given by $\frac{{}^nP_r}{r!}$.

$$\begin{aligned} \frac{{}^nP_r}{r!} &= \frac{\frac{n!}{(n-r)!}}{r!} \\ &= \frac{n!}{(n-r)!} \times \frac{1}{r!} \\ &= \frac{n!}{(n-r)!r!} \end{aligned}$$

${}^nC_r$ can also be written as $\binom{n}{r}$.

EXAMPLE 14

a A bag contains 3 white and 2 black counters labelled W_1, W_2, W_3 and B_1, B_2. If 2 counters are drawn out of the bag, in how many ways can this happen if order is not important?

b If 12 coins are tossed, find the number of ways of tossing 7 tails.

c **i** A committee of 5 people is formed at random from a group of 15 students. In how many different ways can the committee be formed?

ii If the group consists of 9 senior and 6 junior students, in how many ways can the committee be formed if it is to have 3 senior and 2 junior students in it?

d A team of 6 men and 5 women is chosen at random from a group of 10 men and 9 women. If Kaye and Greg both hope to be chosen in the team, find the probability that:

i both will be chosen **ii** neither will be chosen.

Solution

a Possible arrangements (unordered) are:

W_1W_2 W_2W_3 W_3B_1 B_1B_2
W_1W_3 W_2B_1 W_3B_2
W_1B_1 W_2B_2
W_1B_2

There are 10 different combinations.

Using combinations, the number of different arrangements of choosing 2 counters from 5 is ${}^5C_2$.

$$\begin{aligned}{}^5C_2 &= \frac{5!}{(5-2)!2!}\\ &= \frac{5!}{3!2!}\\ &= 10\end{aligned}$$

b The order is not important.

There are ${}^{12}C_7$ ways of tossing 7 tails from 12 coins.

$$\begin{aligned}{}^{12}C_7 &= \frac{12!}{(12-7)!7!}\\ &= \frac{12!}{5!7!}\\ &= 792\end{aligned}$$

Operation	Casio scientific	Sharp scientific
Enter data	12 SHIFT ^nC_r 7 =	12 2ndF ^nC_r 7 =

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c **i** The order of the committee is not important.

Number of arrangements $= \binom{15}{5}$

$= 3003$

Operation	Casio scientific	Sharp scientific
Enter data	15 SHIFT ^nC_r 5 =	15 2ndF ^nC_r 5 =

ii 3 senior students can be chosen in $\binom{9}{3}$ or 84 ways.

2 junior students can be chosen in $\binom{6}{2}$ or 15 ways.

Total number of arrangements $= \binom{9}{3} \times \binom{6}{2}$

$= 84 \times 15$

$= 1260$

d **i** The number of possible teams $= {}^{10}C_6 \times {}^9C_5$

$= 210 \times 126$

$= 26\ 460$

For Kaye to be chosen, then 4 out of the other 8 women will be chosen i.e. ${}^8C_4$.

For Greg to be chosen, 5 out of the other 9 men will be chosen i.e. ${}^9C_5$.

Number of combinations $= {}^8C_4 \times {}^9C_5$

$= 70 \times 126$

$= 8820$

Probability $= \dfrac{8820}{26\ 460}$

$= \dfrac{1}{3}$

ii For Kaye and Greg not to be included, then 5 out of the other 8 women and 6 out of the other 9 men will be chosen.

Number of combinations $= {}^8C_5 \times {}^9C_6$

$= 56 \times 84$

$= 4704$

Probability $= \dfrac{4704}{26\ 460}$

$= \dfrac{8}{45}$

EXT1 Exercise 3.05 Combinations

1 Evaluate:

a $\binom{9}{5}$ **b** $\binom{12}{7}$ **c** $\binom{8}{3}$ **d** ${}^{10}C_4$ **e** ${}^{11}C_5$

2 **a** Evaluate:

i ${}^{10}C_0$ **ii** ${}^{7}C_0$ **iii** $\binom{14}{0}$ **iv** ${}^{9}C_9$ **v** $\binom{11}{11}$

b Hence evaluate:

i ${}^{n}C_0$ **ii** ${}^{n}C_n$

3 Find the number of different ways that a random committee of 6 people can be made from a group of:

a 8 people **b** 9 people **c** 11 people

d 15 people **e** 20 people.

4 **a** A set of 3 red cards and 3 blue cards is placed in a box. By naming the red cards R_1, R_2 and R_3 and the blue cards B_1, B_2 and B_3, list the number of different arrangements possible when 2 cards are drawn out at random, with order not important. How many arrangements are possible?

b If there are 10 red and 10 blue cards and 7 are drawn out at random, how many different combinations are possible?

5 A coin is tossed 20 times. How many different arrangements are there for tossing 5 heads?

6 A set of 10 marbles are placed in a bag and 6 selected at random. In how many different ways can this happen?

7 In poker, 5 cards are dealt from a pack of 52 playing cards. How many different arrangements are possible?

8 Three cards are drawn at random from a set of 10 cards with the numbers 0 to 9 on them. How many different arrangements are possible if order is:

a important **b** unimportant?

9 A debating team of 3 is chosen from a class of 14 students. In how many ways can the team be selected if order is:

a important **b** unimportant?

10 A bag contains 23 lollies. If I take 6 lollies out of the bag, how many different combinations are possible?

11 A team of 4 players is chosen at random from a group of 20 tennis players to play an exhibition match. In how many ways could the team be chosen?

12 A group of 3 students is chosen at random from a class of 27 to go on a student representative council. In how many different ways could this be done?

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13 A board of 8 people is chosen from a membership of 35. How many different combinations are possible?

14 A basketball team of 5 players is selected at random from a group of 12 PE students.

a In how many ways can the team be selected?

b Find the probability that Erik is selected as one of the team members.

c Find the probability that Erik and Jens are both selected.

15 A committee of 6 people is to be selected at random from a group of 11 men and 12 women. Find the number of possible committees if:

a there is no restriction on who is on the committee

b all committee members are to be male

c all committee members are to be female

d there are to be 3 men and 3 women

e Anna is included

f Bruce is not included

g there are to be 4 women and 2 men.

16 A horse race has 15 horses competing. At the TAB, a quinella pays out on the horses that come in first and second, in either order. Ryan decides to bet on all possible combinations of quinellas. If it costs him $1 a bet, how much does he pay?

17 A group of 25 students consist of 11 who play a musical instrument and 14 who do not. Find the number of different arrangements possible if a group of 9 students is selected at random:

a with no restriction

b who all play musical instruments

c where 5 play musical instruments

d where 2 do not play musical instruments.

18 A set of cards consists of 8 yellow and 7 red cards, each showing a different picture.

a If 10 cards are selected at random, find the number of different arrangements possible.

b If 8 cards are selected, find the number of arrangements of selecting:

i 4 yellow cards

ii 6 yellow cards

iii 7 yellow cards

iv 5 red cards.

19 Ten cards are selected at random from a set of 52 playing cards. Find the number of combinations selected if:

a there are no restrictions (answer in scientific notation correct to 3 significant figures)

b they are all hearts

c there are 7 hearts

d they are all red cards

e there are 4 aces.

20 An animal refuge has 17 dogs and 21 cats. If a nursing home orders 12 animals at random to be companion animals, find the number of ways that the order would have:

a 7 dogs **b** 9 dogs **c** 10 dogs

d 4 cats **e** 6 cats.

21 There are 8 white, 9 red and 5 blue marbles in a bag and 7 are drawn out at random. Find the number of arrangements possible:

a with no restriction

b if all marbles are red

c if there are 3 white and 2 red marbles

d if there are 4 red and 1 blue marbles

e if there are 4 white and 2 blue marbles.

22 Out of a group of 25 students, 7 walk to school, 12 catch a train and 6 catch a bus. If 6 students are selected, find the number of combinations if:

a all walk to school

b no-one catches a bus

c 3 walk to school and 1 catches a bus

d 1 walks to school and 4 catch a train

e 3 catch a train and 1 catches a bus.

23 At a karaoke night, a group of 14 friends decide that 4 of them will sing a song together. Of the friends, 5 have previously sung this song before. In how many ways can they do this if they select:

a friends who have all sung the song previously

b 2 of the friends who sang the song previously

c none of the friends who sang the song previously?

24 **a** Evaluate ${}^{12}C_5$.

b Evaluate ${}^{12}C_7$.

c By using factorial notation, show why ${}^{12}C_5 = {}^{12}C_7$.

25 By evaluating both sides, show that ${}^{9}C_6 = {}^{8}C_6 + {}^{8}C_5$.

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26 Show that $\binom{13}{7} = \binom{13}{6}$.

27 Show that $\binom{10}{4} = \binom{9}{4} + \binom{9}{3}$.

28 Prove that $\binom{n}{r} = \binom{n}{n-r}$.

29 Prove that ${}^nP_r = r!\,{}^nC_r$.

30 Prove that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

EXT1 3.06 Pascal's triangle and binomial coefficients

WS Pascal's triangle

WS Pascal's triangle

The binomial expansion

Pascal's triangle

Here is Pascal's triangle. Can you see the patterns in each row and between rows? Could you add the next row using these patterns?

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

INVESTIGATION

COMBINATIONS AND PASCAL'S TRIANGLE

There is a relationship between Pascal's triangle and combinations ${}^nC_r$.

Find ${}^1C_0$, ${}^1C_1$, ${}^2C_0$, ${}^2C_1$, ${}^2C_2$, ${}^3C_0$, ${}^3C_1$, ${}^3C_2$, ${}^3C_3$ and so on. Compare these with Pascal's triangle. Can you see any relationships or patterns?

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The numbers in Pascal's triangle are the same as combinations as shown below.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & {}^0C_0 & & & & \\
 & & & {}^1C_0 & & {}^1C_1 & & & \\
 & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & & \\
 & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 & \\
{}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4
\end{array}$$

You can use combinations to complete further rows in Pascal's triangle.

EXAMPLE 15

Add the next 2 lines to Pascal's triangle.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Solution

The 1st row after this is ${}^5C_0 = 1$, ${}^5C_1 = 5$, ${}^5C_2 = 10$, ${}^5C_3 = 10$, ${}^5C_4 = 5$, ${}^5C_5 = 1$

The 2nd row after this is ${}^6C_0 = 1$, ${}^6C_1 = 6$, ${}^6C_2 = 15$, ${}^6C_3 = 20$, ${}^6C_4 = 15$, ${}^6C_5 = 6$, ${}^6C_6 = 1$

So Pascal's triangle becomes:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

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There is a relationship between Pascal's triangle and the expansions of binomial products $(x+y)^n$.

EXAMPLE 16

Expand each binomial product:

a $(x+y)^0$ **b** $(x+y)^1$ **c** $(x+y)^2$ **d** $(x+y)^3$ **e** $(x+y)^4$

Solution

a $(x+y)^0 = 1$ since $a^0 = 1$

b $(x+y)^1 = x + y$ since $a^1 = a$

c $(x+y)^2 = x^2 + 2xy + y^2$ perfect square

d
$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(x^2 + 2xy + y^2)\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

e
$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)^3\\ &= (x+y)(x^3 + 3x^2y + 3xy^2 + y^3)\\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3 + x^3y + 3x^2y^2 + 3xy^3 + y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

Look at the coefficients of each of the **binomial expansions** in the example.
The coefficients of each term in these binomial expansions form Pascal's triangle.

$(x+y)^0$					1				
$(x+y)^1$				1		1			
$(x+y)^2$			1		2		1		
$(x+y)^3$		1		3		3		1	
$(x+y)^4$	1		4		6		4		1

Since the numbers in Pascal's triangle are also combinations, this means that the coefficients in the binomial expansion of $(x + y)^n$ can be written as combinations.

Binomial expansion

$$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + \ldots + {}^nC_kx^{n-k}y^k + \ldots + {}^nC_ny^n$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + \binom{n}{n}y^n$$

EXAMPLE 17

Use ${}^nC_r$ to expand each binomial product.

a $(p + q)^2$ **b** $(x + y)^3$ **c** $(a + b)^5$

Solution

a
$$\begin{aligned}(p + q)^2 &= {}^2C_0p^2q^0 + {}^2C_1p^1q^1 + {}^2C_2p^0q^2\\ &= 1p^2 + 2pq + 1q^2\\ &= p^2 + 2pq + q^2\end{aligned}$$

This agrees with the formula for a perfect square

b
$$\begin{aligned}(x + y)^3 &= {}^3C_0x^3y^0 + {}^3C_1x^2y^1 + {}^3C_2x^1y^2 + {}^3C_3x^0y^3\\ &= 1x^3 + 3x^2y + 3xy^2 + 1y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

c
$$\begin{aligned}(a + b)^5 &= {}^5C_0a^5 + {}^5C_1a^4b + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + {}^5C_5b^5\\ &= 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5\\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

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We can use this formula to expand other binomial products.

EXAMPLE 18

a Expand $(4 + x)^5$.

b Expand $(3x - 2)^4$.

c If $(1 + \sqrt{3})^4 = a + b\sqrt{3}$, evaluate a and b.

Solution

a $(4 + x)^5 = {}^5C_0 4^5 + {}^5C_1 4^4 x + {}^5C_2 4^3 x^2 + {}^5C_3 4^2 x^3 + {}^5C_4 4x^4 + {}^5C_5 x^5$

$= 1(1024) + 5(256)x + 10(64)x^2 + 10(16)x^3 + 5(4)x^4 + (1)x^5$

$= 1024 + 1280x + 640x^2 + 160x^3 + 20x^4 + x^5$

b $(3x - 2)^4 = {}^4C_0(3x)^4 + {}^4C_1(3x)^3(-2) + {}^4C_2(3x)^2(-2)^2 + {}^4C_3(3x)(-2)^3 + {}^4C_4(-2)^4$

$= 1(81x^4) + 4(27x^3)(-2) + 6(9x^2)(4) + 4(3x)(-8) + (1)16$

$= 81x^4 - 216x^3 + 216x^2 - 96x + 16$

c Expand $(1 + \sqrt{3})^4$.

$(1 + \sqrt{3})^4 = {}^4C_0 1^4 + {}^4C_1 1^3(\sqrt{3}) + {}^4C_2 1^2(\sqrt{3})^2 + {}^4C_3 1^1(\sqrt{3})^3 + {}^4C_4(\sqrt{3})^4$

$= 1 + 4\sqrt{3} + 6(\sqrt{3})^2 + 4(\sqrt{3})^3 + (\sqrt{3})^4$

$= 1 + 4\sqrt{3} + 6\sqrt{9} + 4\sqrt{27} + \sqrt{81}$

$= 1 + 4\sqrt{3} + 18 + 4 \times 3\sqrt{3} + 9$

$= 28 + 16\sqrt{3}$

So $a = 28$ and $b = 16$.

Properties of coefficients

The patterns and symmetry of Pascal's triangle show some properties of combinations and binomial expansions. We can prove these by using the definition ${}^nC_r = \frac{n!}{r!(n-r)!}$ that you saw in the previous secton.

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Since the first and last values in each row of Pascal's triangle are 1, we have the property:

$$^nC_0 = {}^nC_n = 1$$

Proof

$$^nC_r = \frac{n!}{r!(n-r)!}$$

$$\begin{aligned} ^nC_n &= \frac{n!}{n!(n-n)!} \\ &= \frac{n!}{n!0!} \\ &= \frac{n!}{n!} \\ &= 1 \end{aligned}$$

$$\begin{aligned} ^nC_0 &= \frac{n!}{0!(n-0)!} \\ &= \frac{n!}{0!n!} \\ &= 1 \end{aligned}$$

By symmetry of Pascal's triangle:

$$^nC_k = {}^nC_{n-k}$$

Proof

$$^nC_k = \frac{n!}{k!(n-k)!}$$

$$\begin{aligned} ^nC_{n-k} &= \frac{n!}{(n-k)!(n-[n-k])!} \\ &= \frac{n!}{(n-k)!k!} \end{aligned}$$

$$\therefore {}^nC_k = {}^nC_{n-k}$$

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Since each number in Pascal's triangle is the sum of the 2 numbers in the row above it:

Pascal's triangle identity

$$^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$$

Proof

$$^{n}C_{k} = \frac{n!}{k!(n-k)!}$$

$$\begin{aligned}
{}^{n-1}C_{k-1} + {}^{n-1}C_{k} &= \frac{(n-1)!}{(k-1)!([n-1]-[k-1])!} + \frac{(n-1)!}{k!([n-1]-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(n-1)!(k+n-k)}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

$$\therefore {}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$$

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EXAMPLE 19

Show that:

a $^8C_3 = {^8C_5}$

b $\binom{7}{4} = \binom{6}{3} + \binom{6}{4}$

Solution

a

$$^8C_3 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!}$$

$$^8C_5 = \frac{8!}{(8-5)!5!} = \frac{8!}{3!5!}$$

So $^8C_3 = {^8C_5}$.

b

$$\text{LHS} = \binom{7}{4} = \frac{7!}{(7-4)!4!} = \frac{7!}{3!4!}$$

$$\begin{aligned}
\text{RHS} &= \binom{6}{3} + \binom{6}{4} \\
&= \frac{6!}{(6-3)!3!} + \frac{6!}{(6-4)!4!} \\
&= \frac{6!}{3!3!} + \frac{6!}{2!4!} \\
&= \frac{6! \times 4}{3!3! \times 4} + \frac{6! \times 3}{2!4! \times 3} \\
&= \frac{6! \times 4}{4!3!} + \frac{6! \times 3}{4!3!} \\
&= \frac{4(6!) + 3(6!)}{4!3!} \\
&= \frac{7(6!)}{4!3!} \\
&= \frac{7!}{4!3!} \\
&= \text{LHS}
\end{aligned}$$

So $\binom{7}{4} = \binom{6}{3} + \binom{6}{4}$.

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INVESTIGATION

FIBONACCI AND OTHER SEQUENCES

Pascal's triangle is not the only interesting pattern of numbers that has practical uses. The Fibonacci sequence is also very interesting and can be seen in nature. Research Fibonacci and the golden ratio.

Discover how the number phi (ϕ) is related to Fibonacci, trigonometry and the number π.

Can you find any other interesting number patterns or sequences?

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EXT1 Exercise 3.06 Pascal's triangle and binomial coefficients

1 Show that:

a $\binom{9}{5} = \binom{9}{4}$ **b** ${}^7C_2 = {}^7C_5$ **c** $\binom{12}{5} = \binom{12}{7}$

2 Prove that:

a ${}^7C_5 = {}^6C_4 + {}^6C_5$ **b** $\binom{10}{6} = \binom{9}{5} + \binom{9}{6}$ **c** $\binom{7}{3} = \binom{6}{2} + \binom{6}{3}$

3 Show that $\binom{n}{1} = \binom{n}{n-1}$.

4 Use the symmetry of Pascal's triangle to find x if ${}^7C_x = {}^7C_2$.

5 If $\binom{12}{3} = \binom{12}{y}$, use the symmetry of Pascal's triangle to find y.

6 Find the value of a if ${}^{11}C_a = {}^{11}C_8$.

7 Use Pascal's triangle identity to find n if $\binom{n}{6} = \binom{10}{5} + \binom{10}{6}$.

8 Use Pascal's triangle identity to find k if ${}^{20}C_7 = {}^{19}C_k + {}^{19}C_7$.

9 Expand each binomial product.

a $(a + x)^4$ **b** $(a + x)^6$ **c** $(a + x)^5$

d $(2a + 1)^3$ **e** $(x - 2)^7$ **f** $(4x^2 + 3)^4$

g $(3 - 2x)^6$ **h** $(4a - 5b)^3$ **i** $(2 + 3m)^5$

j $(1 - 2x)^8$

10 Expand:

a $\left(\sqrt{2}+1\right)^5$ **b** $\left(\sqrt{3}-1\right)^6$ **c** $\left(\sqrt{3}+\sqrt{5}\right)^4$

d $\left(3+\frac{x}{2}\right)^4$ **e** $\left(x+\frac{1}{x}\right)^5$ **f** $\left(1-\frac{x}{2}\right)^3$

g $\left(\frac{a}{3}-\frac{b}{2}\right)^3$

11 Evaluate a and b if $\left(\sqrt{2}+3\right)^3 = a + b\sqrt{2}$.

12 If $\left(2-\sqrt{5}\right)^4 = a + b\sqrt{5}$, evaluate a and b.

13 Evaluate x and y if $\left(\sqrt{3}-1\right)^5 = x + \sqrt{y}$.

14 If $\left(\sqrt{2}+\sqrt{3}\right)^3 = a\sqrt{2} + b\sqrt{3}$, evaluate a and b.

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3. TEST YOURSELF

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 A group of 4 people sit together in a bus with 2 seats facing forwards and 2 facing backwards. If one person cannot sit facing backwards, in how many ways can the 4 people be arranged?

A 36 **B** 24 **C** 12 **D** 48

2 Which one of these formulas is correct?

A ${}^nC_k = {}^nC_{k-1}$ **B** ${}^nC_k = {}^{n+1}C_{k+1} + {}^{n+1}C_k$

C ${}^{n-1}C_k = {}^nC_{k-1} + {}^nC_k$ **D** ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

3 Find the smallest number of balls chosen from a bag containing yellow, white, blue, black and green balls so that 2 must be the same colour.

A 5 **B** 6 **C** 7 **D** 4

4 Expand $(x - 3)^5$.

5 12 people are to be seated around a table.

a In how many ways can they be seated?

b In how many ways can they be seated if 2 particular people are not to sit together?

c Find the probability that 2 friends will be seated together.

6 Show that:

a ${}^{11}C_3 = {}^{11}C_8$ **b** ${}^{10}C_1 = {}^{10}C_9$

c $\binom{9}{7} = \binom{8}{6} + \binom{8}{7}$ **d** $\binom{11}{6} = \binom{10}{5} + \binom{10}{6}$

7 A committee of 5 people is chosen at random from a group of 10 women and 12 men. Find the number of ways in which the committee could be formed with:

a no restriction on how many men or women are on the committee

b a committee of 2 men and 3 women.

8 Expand $(2x + 3y)^4$.

9 Any one person has up to 150 000 hairs on their head. A city has a population of 256 840. Show that there are at least 2 people in this city that have exactly the same number of hairs on their head.

10 In how many ways can the letters of the word AUSTRALIA be arranged?

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11 Find the number of ways of arranging 7 people:

a in a straight line **b** in a circle.

12 Expand:

a $(\sqrt{5}-2)^6$ **b** $\left(x-\frac{3}{x^2}\right)^4$ **c** $(2-3\sqrt{2})^7$

13 A set of 10 cards, numbered 1 to 10, is placed into a box and 3 drawn out at random. Find the number of arrangements possible if order is:

a important **b** unimportant.

14 A group of 10 boys and 16 girls are on a school excursion. Five of them are chosen at random to help the teacher check the rolls. Find the number of ways these students could be selected if:

a there is no restriction

b 3 are to be girls

c all are to be boys.

15 If $(\sqrt{5}-\sqrt{3})^5 = a\sqrt{5}+b\sqrt{3}$, evaluate a and b.

16 In how many ways can 3-letter combinations be selected at random from the word RANDOM?

17 How many different arrangements are there of the letters in the word:

a PERMUTATION **b** COMBINATION **c** FACTORIAL

d PROBABILITY **e** SELECTION?

18 A batch of 750 bricks is sorted into 7 different categories according to their colour, density and other properties. Find the smallest number of bricks that will be in at least one of the categories.

19 Evaluate:

a $\binom{6}{4}$ **b** ${}^9P_7$

20 A serial number is made up of 4 letters and 2 numbers. If 0 is not allowed, find how many serial numbers are possible.

21 a Evaluate 0!

b Show that $\binom{n}{0}=\binom{n}{n}$.

22 Show that if 33 points are drawn inside a square with sides 4 cm, at least 3 of these points can join to give a triangle with an area less than 1 cm^2.

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3. CHALLENGE EXERCISE

1 A bag contains 8 cards, each with a different number from 1 to 8. If you select 5 numbers at random, show that at least 2 of the numbers add up to 9.

2 If a computer randomly generates 4-letter 'words' from the letters in MATHEMATICS, find the probability that the word made is CAME.

3 Simplify $\frac{{}^nC_k}{{}^nC_{k-1}}$.

4 Numbers are formed from the digits 1, 2, 3, 3 and 7 at random.

a In how many ways can they be arranged with no restrictions?

b In how many ways can they be arranged to form a number greater than 30 000?

5 A charm bracelet has 6 different charms on it. In how many ways can the charms be arranged if the bracelet:

a has a clasp? **b** has no clasp?

6 A management committee is made up of 5 athletes and 3 managers. If the committee is formed from a group of 20 athletes and 10 managers at random, find:

a the number of different ways in which the committee could be formed

b the probability that Patrick, an athlete, is included

c the probability that both Patrick and his sister Alexis, who is a manager, are included

d the probability that Patrick and Alexis are excluded from the committee.

7 A group of n people sit around a circular table.

a In how many ways can they be arranged?

b How many arrangements are possible if k people sit together?

8 By writing 0.99 as $1 - 0.01$ and expanding $(1 - 0.01)^3$, evaluate 0.99^3 to 4 decimal places.

9 An equilateral triangle has sides 3 cm. If 10 points are randomly drawn inside the triangle, show that there are at least 2 points whose distance apart is less than or equal to 1 cm.

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