FUNCTIONS

10.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this chapter you will study the definition and laws of logarithms and their relationship with the exponential and logarithmic functions. You will meet a new irrational number, *e*, that has special properties, solve exponential and logarithmic equations, and examine applications of exponential and logarithmic functions, including exponential growth and decay.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- graph exponential and logarithmic functions
- understand and use Euler's number, e
- differentiate exponential functions
- convert between exponential and logarithmic forms using the definition of a logarithm
- identify and apply logarithm laws
- solve exponential equations using logarithms
- solve practical formulas involving exponents and logarithms
- EXT1 understand and solve problems involving exponential growth and decay

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TERMINOLOGY

Euler's number: This number, e, approximately 2.718 28, is an important constant that is the base of natural logarithms

EXT1 **exponential decay**: When a quantity decreases according to the exponential function $N = Ae^{kt}$, where k is negative

EXT1 **exponential growth**: When a quantity increases according to the exponential function $N = Ae^{kt}$, where k is positive

exponential function : A function in the form $y = a^x$

logarithm: The logarithm of a positive number y is the power to which a given number a, called the base, must be raised in order to produce the number y, so $\log_a y = x$ means $y = a^x$

logarithmic function: A function in the form $y = \log_a x$

WS Graphing exponentials

WS Exponential functions

WS Translating exponential graphs

10.01 Exponential functions

An **exponential function** is in the form $y = a^x$, where $a > 0$.

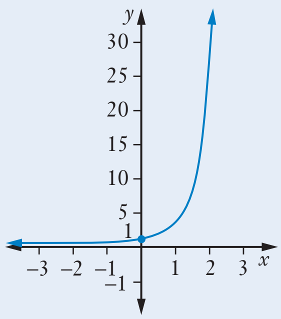
EXAMPLE 1

Sketch the graph of the function $y = 5^x$ and state its domain and range.

Solution

Complete a table of values for $y = 5^x$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125



Notice that a^x is always positive. So there is no x-intercept and $y > 0$.

For the y-intercept, when $x = 0$, $y = 5^0 = 1$.

The y-intercept is 1.

From the graph, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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INVESTIGATION

THE VALUE OF a IN $y = a^x$

Notice that the exponential function $y = a^x$ is only defined for $a > 0$.

1 Suppose $a = 0$. What would the function $y = 0^x$ look like? Try completing a table of values or use technology to sketch the graph. Is the function defined for positive values of x, negative values of x or when $x = 0$? What if x is a fraction?

2 Suppose $a < 0$. What would the function $y = (-2)^x$ look like?

3 For $y = 0^x$ and $y = (-2)^x$:

a is it possible to graph these functions at all?

b are there any discontinuities on the graphs?

c do they have a domain and range?

The exponential function $y = a^x$

- Domain $(-\infty, \infty)$, range $(0, \infty)$.
- The y-intercept ($x = 0$) is always 1 because $a^0 = 1$.
- The graph is always above the x-axis and there is no x-intercept ($y = 0$) because $a^x > 0$ for all values of x.
- The x-axis is an **asymptote**.

EXAMPLE 2

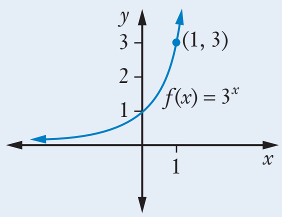
Sketch the graph of:

a $f(x) = 3^x$ **b** $y = 2^x + 1$

Solution

a The curve is above the x-axis with y-intercept 1. We must show another point on the curve.

$f(1) = 3^1 = 3$.



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b For y-intercept, $x = 0$:

$y = 2^0 + 1$

$= 1 + 1$

$= 2$

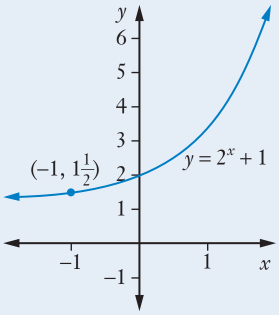
Find another point, when $x = -1$:

$y = 2^{-1} + 1$

$= \frac{1}{2} + 1$

$= 1\frac{1}{2}$

Notice that this is the graph of $y = 2^x$ moved up 1 unit.



EXAMPLE 3

Sketch the graph of:

a $f(x) = 3(4^x)$ **b** $y = 2^{x+1}$

Solution

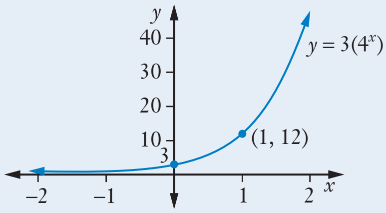
a The values of $f(x) = 3(4^x)$ will be 3 times greater than 4^x so its curve will be steeper.

$f(-1) = 3(4^{-1}) = 0.75$

$f(0) = 3(4^0) = 3$

$f(1) = 3(4^1) = 12$

$f(2) = 3(4^2) = 48$

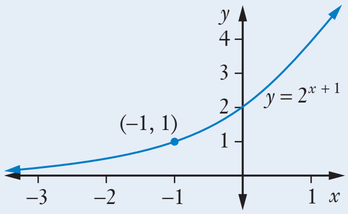


b $f(-1) = 2^{-1+1} = 1$

$f(0) = 2^{0+1} = 2$

$f(1) = 2^{1+1} = 4$

$f(2) = 2^{2+1} = 8$



This is the graph of $y = 2^x$ shifted 1 unit left.

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Reflections of exponential functions

We can reflect the graph of $y = a^x$ using what we learned in Chapter 7, *Further functions*.

EXAMPLE 4

Given $f(x) = 3^x$, sketch the graph of:

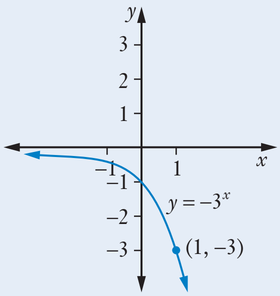
a $y = -3^x$ **b** $y = 3^{-x}$ **c** $y = -3^{-x}$

Solution

a Given $f(x) = 3^x$, then $y = -f(x) = -3^x$.

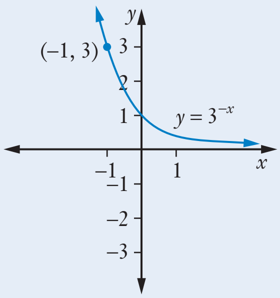
This is a reflection of $f(x)$ in the x-axis.

Note: -3^x means $-(3^x)$, *not* $(-3)^x$.



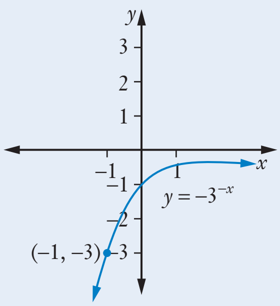
b Given $f(x) = 3^x$, then $y = f(-x) = 3^{-x}$.

This is a reflection of $f(x)$ in the y-axis.



c Given $f(x) = 3^x$, then $y = -f(-x) = -3^{-x}$.

This is a reflection of $f(x)$ in both the x- and y-axes.



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INVESTIGATION

GRAPHS OF EXPONENTIAL FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of the exponential functions below. Look for similarities and differences within each set.

a $y = 2^x, y = 2^x + 1, y = 2^x + 3, y = 2^x - 5$

b $y = 3(2^x), y = 4(2^x), y = -2^x, y = -3(2^x)$

c $y = 3(2^x) + 1, y = 4(2^x) + 3, y = -2^x + 1, y = -3(2^x) - 3$

d $y = 2^{x+1}, y = 2^{x+2}, y = 2^{x-1}, y = 2^{x-3}, y = 2^{-x}$

e $y = 2^{-x}, y = 2\left(2^{-x}\right), y = -2^{-x}, y = -3\left(2^{-x}\right), y = 2^{-x-1}$

Exercise 10.01 Exponential functions

1 Sketch each exponential function.

a $y = 2^x$ **b** $y = 4^x$ **c** $f(x) = 3^x + 2$ **d** $y = 2^x - 1$

e $f(x) = 3(2^x)$ **f** $y = 4^{x+1}$ **g** $y = 3(4^{2x}) - 1$ **h** $f(x) = -2^x$

i $y = 2(4^{-x})$ **j** $f(x) = -3(5^{-x}) + 4$

2 State the domain and range of each function.

a $f(x) = 2^x$ **b** $y = 3^x + 5$ **c** $f(x) = 10^{-x}$ **d** $f(x) = -5^x + 1$

3 Given $f(x) = 2^x$ and $g(x) = 3x - 4$, find:

a $f(g(x))$ **b** $g(f(x))$

4 **a** Sketch the graph of $f(x) = 4(3^x) + 1$.

b Sketch the graph of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5 Sales numbers, N, of a new solar battery are growing over t years according to the formula $N = 450(3^{0.9t})$.

a Draw a graph of this function.

b Find the initial number of sales when $t = 0$.

c Find the number of sales after:

i 3 years

ii 5 years

iii 10 years

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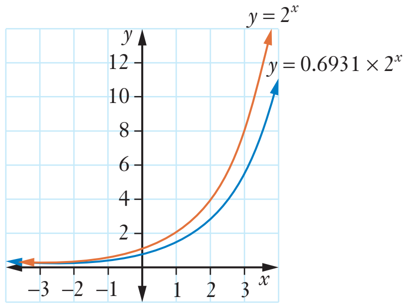
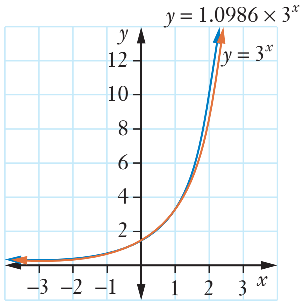
10.02 Euler's number, *e*

The gradient function of exponential functions is interesting. Notice that the gradient of an exponential function is always increasing, and increases at an increasing rate.

If you sketch the derivative function of an exponential function, then it too is an exponential function! Here are the graphs of the derivative functions (in blue) of $y = 2^x$ and $y = 3^x$ (in red) together with their equations.





Notice that the graph of the derivative function of $y = 3^x$ is very close to the graph of the original function.

We can find a number close to 3 that gives exactly the same derivative function as the original graph. This number is approximately 2.718 28, and is called **Euler's number**, *e*. Like π, the number *e* is irrational.

Euler's number

$$e \approx 2.718\,28$$

DID YOU KNOW?

Leonhard Euler

Like π, Euler's number, *e*, is a **transcendental** number, which is an irrational number that is not a surd. This was proven by a French mathematician, **Charles Hermite**, in 1873. The Swiss mathematician **Leonhard Euler** (1707–83) gave *e* its symbol, and he gave an approximation of *e* to 23 decimal places. Now *e* has been calculated to over a trillion decimal places.

Euler gave mathematics much of its important notation. He caused π to become standard notation for pi and used *i* for the square root of −1. He also introduced the symbol Σ for sums and $f(x)$ notation for functions.

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EXAMPLE 5

Sketch the graph of the exponential function $y = e^x$.

Solution

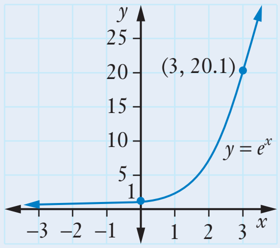
Use e^x on your calculator to draw up a table of values. For example, to calculate e^{-3}:

Casio scientific	Sharp scientific
SHIFT e^x (−) 3 =	2ndF e^x +/− 3 =

$e^{-3} = 0.049\,78\ldots$

x	−3	−2	−1	0	1	2	3
y	0.05	0.1	0.4	1	2.7	7.4	20.1

(rounded figures)



EXAMPLE 6

The salmon population in a river over time can be described by the exponential function $P = 200e^{0.3t}$ where t is time in years.

a Find the population after 3 years.

b Draw the graph of the population.

Solution

a $P = 200e^{0.3t}$

When $t = 3$:

$P = 200e^{0.3 \times 3}$

$= 491.9206\ldots$

≈ 492

So after 3 years there are 492 salmon.

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b The graph is an exponential curve. Finding some points will help us graph it accurately.

When $t = 0$: $P = 200e^{0.3 \times 0}$

$= 200$ This is also the P-intercept.

When $t = 1$: $P = 200e^{0.3 \times 1}$

$= 269.9717\ldots$

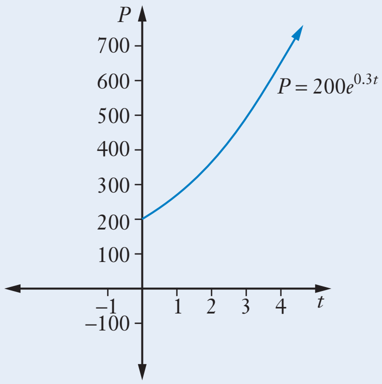
≈ 270

When $t = 2$: $P = 200e^{0.3 \times 2}$

$= 364.4237\ldots$

≈ 364

We already know $P \approx 492$ when $t = 3$.



Time, $t \geq 0$, so don't sketch the curve for negative values of t.

Exercise 10.02 Euler's number, *e*

1 Sketch the curve $f(x) = 2e^{x-2}$.

2 Evaluate, correct to 2 decimal places:

a $e^{1.5}$ **b** e^{-2} **c** $2e^{0.3}$ **d** $\dfrac{1}{e^3}$ **e** $-3e^{-3.1}$

3 Sketch each exponential function.

a $y = 2e^x$ **b** $f(x) = e^x + 1$ **c** $y = -e^x$ **d** $y = e^{-x}$ **e** $y = -e^{-x}$

4 State the domain and range of $f(x) = e^x - 2$.

5 If $f(x) = e^x$ and $g(x) = x^3 + 3$, find:

a $f(g(x))$ **b** $g(f(x))$

6 The volume V of a metal in mm^3 expands as it is heated over time according to the formula $V = 25e^{0.7t}$, where t is in minutes.

a Sketch the graph of $V = 25e^{0.7t}$.

b Find the volume of the metal at:

i 3 minutes **ii** 8 minutes

c Is this formula a good model for the rise in volume? Why?

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7 The mass of a radioactive substance in g is given by $M = 150e^{-0.014t}$ where t is in years. Find the mass after:

a 10 years **b** 50 years **c** 250 years

8 The number of koalas in a forest is declining according to the formula $N = 873e^{-0.078t}$ where t is the time in years.

a Sketch a graph showing this decline in numbers of koalas for the first 6 years.

b Find the number of koalas:

i initially **ii** after 5 years **iii** after 10 years

Photo courtesy Margaret Grove

9 An object is cooling down according to the exponential function $T = 23 + 125e^{-0.06t}$ where T is the temperature in °C and t is time in minutes.

a Find the initial temperature.

b Find the temperature at:

i 2 minutes **ii** 5 minutes **iii** 10 minutes **iv** 2 hours

c What temperature is the object tending towards? Can you explain why?

10 A population is growing exponentially. If the initial population is 20 000 and after 5 years the population is 80 000, draw a graph showing this information.

11 The temperature of a piece of iron in a smelter is 1000°C and it is cooling down exponentially. After 10 minutes the temperature is 650°C. Draw a graph showing this information.

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10.03 Differentiation of exponential functions

WS Differentiating exponential functions

Euler's number, e, is the special number such that the derivative function of $y = e^x$ is itself. The derivative of e^x is e^x.

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 7

a Differentiate $y = e^x - 5x^2$.

b Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

Solution

a $\frac{dy}{dx} = e^x - 10x$

b Gradient of the tangent:

$$\frac{dy}{dx} = e^x$$

At $(1, e)$:

$$\begin{aligned}\frac{dy}{dx} &= e^1 \\ &= e\end{aligned}$$

So $m = e$

Equation:

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ y - e &= e(x - 1) \\ &= ex - e \\ y &= ex\end{aligned}$$

The rule for differentiating $kf(x)$ works with the rule for e^x as well.

Derivative of ke^x

$$\frac{d}{dx}(ke^x) = ke^x$$

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EXAMPLE 8

a Differentiate $y = 5e^x$.

b Find the gradient of the normal to the curve $y = 3e^x$ at the point (0, 3).

Solution

a $\frac{dy}{dx} = 5e^x$

b Gradient of tangent:

$$\frac{dy}{dx} = 3e^x$$

At (0, 3):

$$\begin{aligned}\frac{dy}{dx} &= 3e^0 \\ &= 3 \text{ since } e^0 = 1\end{aligned}$$

So $m_1 = 3$

For normal:

$$\begin{aligned}m_1 m_2 &= -1 \\ 3m_2 &= -1 \\ m_2 &= -\frac{1}{3}\end{aligned}$$

So the gradient of the normal at (0, 3) is $-\frac{1}{3}$.

We can also use other differentiation rules such as the chain rule, product rule and quotient rule with the exponential function.

EXAMPLE 9

Differentiate:

a $y = e^{9x}$

b $y = e^{-5x}$

Solution

a Let $u = 9x$

Then $\frac{du}{dx} = 9$

$$\begin{aligned}y &= e^u \\ \frac{dy}{du} &= e^u \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times 9 \\ &= 9e^u \\ &= 9e^{9x}\end{aligned}$$

b Let $u = -5x$

Then $\frac{du}{dx} = -5$

$$\begin{aligned}y &= e^u \\ \frac{dy}{du} &= e^u \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times (-5) \\ &= -5e^u \\ &= -5e^{-5x}\end{aligned}$$

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The derivative of e^{ax}

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

Proof

Let $u = ax$

Then $\frac{du}{dx} = a$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times a \\ &= ae^u \\ &= ae^{ax}\end{aligned}$$

EXAMPLE 10

Differentiate:

a $y = (1 + e^x)^3$

b $y = \dfrac{2x+3}{e^x}$

Solution

a
$$\begin{aligned}\frac{dy}{dx} &= 3(1 + e^x)^2 \times e^x \\ &= 3e^x(1 + e^x)^2\end{aligned}$$

b
$$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - v'u}{v^2} \\ &= \frac{2e^x - e^x(2x+3)}{(e^x)^2} \\ &= \frac{2e^x - 2xe^x - 3e^x}{e^{2x}} \\ &= \frac{-e^x - 2xe^x}{e^{2x}} \\ &= \frac{-e^x(1+2x)}{e^{2x}} \\ &= \frac{-(1+2x)}{e^x}\end{aligned}$$

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Exercise 10.03 Differentiation of exponential functions

1 Differentiate:

a $y = 9e^x$ **b** $y = -e^x$ **c** $y = e^x + x^2$

d $y = 2x^3 - 3x^2 + 5x - e^x$ **e** $y = (e^x + 1)^3$ **f** $y = (e^x + 5)^7$

g $y = (2e^x - 3)^2$ **h** $y = xe^x$ **i** $y = \dfrac{e^x}{x}$

j $y = x^2e^x$ **k** $y = e^x(2x + 1)$ **l** $y = \dfrac{e^x}{7x - 3}$

m $y = \dfrac{5x}{e^x}$

2 Find the derivative of:

a $y = e^{2x}$ **b** $y = e^{-x}$ **c** $y = 2e^{3x}$

d $y = -e^{7x}$ **e** $y = -3e^{2x} + x^2$ **f** $y = e^{2x} - e^{-2x}$

g $y = 5e^{-x} - 3x + 2$ **h** $y = xe^{4x}$ **i** $y = \dfrac{2e^{3x} - 3}{x + 1}$

j $y = (9e^{3x} + 2)^5$

3 If $f(x) = x^3 + 3x - e^x$, find $f'(1)$ in terms of e.

4 Find the exact gradient of the tangent to the curve $y = e^x$ at the point $(1, e)$.

5 Find the exact gradient of the normal to the curve $y = e^{2x}$ at the point where $x = 5$.

6 Find the gradient of the tangent to the curve $y = 4e^x$ at the point where $x = 1.6$ correct to 2 decimal places.

7 Find the equation of the tangent to the curve $y = -e^x$ at the point $(1, -e)$.

8 Find the equation of the normal to the curve $y = e^{-x}$ at the point where $x = 3$ in exact form.

9 A population P of insects over time t weeks is given by $P = 3e^{1.4t} + 12\ 569$.

a What is the initial population?

b Find the rate of change in the number of insects after:

i 3 weeks **ii** 7 weeks

10 The displacement of a particle over time t seconds is given by $x = 2e^{4t}$ m.

a What is the initial displacement?

b What is the exact velocity after 10 s?

c Find the acceleration after 2 s correct to 1 decimal place.

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11 The displacement of an object in cm over time t seconds is given by $x = 6e^{-0.34t} - 5$. Find:

a the initial displacement

b the initial velocity

c the displacement after 4 s

d the velocity after 9 s

e the acceleration after 2 s

12 The volume V of a balloon in mm^3 as it expands over time t seconds is given by $V = 3e^{0.8t}$.

a Find the volume of the balloon at:

i 3 s **ii** 5 s

b Find the rate at which the volume is increasing at:

i 3 s **ii** 5 s

13 The population of a city is changing over t years according to the formula $P = 34\ 500e^{0.025t}$.

a Find (to the nearest whole number) the population after:

i 5 years **ii** 10 years

b Find the rate at which the population is changing after:

i 5 years **ii** 10 years

14 The depth of water (in metres) in a dam is decreasing over t months according to the formula $D = 3e^{-0.017t}$.

a Find correct to 2 decimal places the depth after:

i 1 month **ii** 2 months **iii** 3 months

b Find correct to 3 decimal places the rate at which the depth is changing after:

i 1 month **ii** 2 months **iii** 3 months

10.04 Logarithms

WS Logarithms

Logarithms

The **logarithm** of a positive number, y, is the **power** to which a **base**, a, must be raised in order to produce the number y. For example, $\log_2 8 = 3$ because $2^3 = 8$.

If $y = a^x$ then x is called the **logarithm of y to the base a**.

Just as the exponential function $y = a^x$ is defined for positive bases only ($a > 0$), logarithms are also defined for $a > 0$. Furthermore, $a \neq 1$ because $1^x = 1$ for all values of x.

Logarithms

$$\text{If } y = a^x \text{ then } x = \log_a y \qquad (a > 0, a \neq 1, y > 0)$$

Logarithms are related to exponential functions and allow us to solve equations like $2^x = 5$.

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EXAMPLE 11

a Write $\log_4 x = 3$ in index form and solve for x.

b Write $5^2 = 25$ in logarithm form.

c Solve $\log_x 36 = 2$.

d Evaluate $\log_3 81$.

e Find the value of $\log_2 \frac{1}{4}$.

Solution

a $\log_a y = x$ means $y = a^x$

$\log_4 x = 3$ means $x = 4^3$

So $x = 64$

b $y = a^x$ means $\log_a y = x$

So $25 = 5^2$ means $\log_5 25 = 2$

c $\log_x 36 = 2$ means $36 = x^2$

$$\begin{aligned} x &= \sqrt{36} \\ &= 6 \end{aligned}$$

Note: x is the base, so $x > 0$.

d $\log_3 81 = x$ means $81 = 3^x$

Solving $3^x = 81$:

$3^x = 3^4$

So $x = 4$

$\log_3 81 = 4$

e Let $\log_2 \frac{1}{4} = x$

Then
$$\begin{aligned} 2^x &= \frac{1}{4} \\ &= \frac{1}{2^2} \\ &= 2^{-2} \\ \therefore x &= -2 \end{aligned}$$

So $\log_2 \frac{1}{4} = -2$

EXAMPLE 12

Simplify:

a $\log_8 1$ **b** $\log_8 8$ **c** $\log_8 8^3$

d $\log_a a^x$ **e** $3^{\log_3 7}$ **f** $a^{\log_a x}$

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Solution

a $\log_8 1 = 0$ because $8^0 = 1$

b $\log_8 8 = 1$ because $8^1 = 8$

c $\log_8 8^3 = 3$ because $8^3 = 8^3$

d $\log_a a^x = x$ because $a^x = a^x$

e Let $\log_3 7 = y$

Then $3^y = 7$

So substituting for y:

$3^{\log_3 7} = 7$

f Let $\log_a x = y$

Then $a^y = x$

So substituting for y:

$a^{\log_a x} = x$

Notice that logarithms and exponentials are inverse operations.

Properties of logarithms

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

Common logarithms and natural logarithms

There are 2 types of logarithms that you can find on your calculator.

- **Common logarithms (base 10)**: $\log_{10} x$ or $\log x$
- **Natural (Naperian) logarithms (base *e*)**: $\log_e x$ or $\ln x$

EXAMPLE 13

a Find $\log_{10} 5.3$ correct to 1 decimal place.

b Evaluate $\log_e 80$ correct to 3 significant figures.

c Loudness in decibels is given by the formula $L = 10 \log_{10}\left(\dfrac{I}{I_0}\right)$ where I_0 is threshold sound, or sound that can barely be heard. Sound louder than 85 decibels can cause hearing damage.

i The loudness of a vacuum cleaner is 10 000 000 times the threshold level, or $10\ 000\ 000I_0$. How many decibels is this?

ii If the loudness of the sound of rustling leaves is 20 dB, find its loudness in terms of I_0.

Solution

a $\log_{10} 5.3 = 0.724\ 2\ \ldots$ log 5.3 =

≈ 0.7

b $\log_e 80 = 4.382\ 0\ldots$ ln 80 =

≈ 4.38

c **i** $L = 10\log_{10}\left(\dfrac{10\,000\,000 I_0}{I_0}\right)$

$= 10\log_{10}(10\ 000\ 000)$

$= 10 \times 7$

$= 70$

So the loudness of the vacuum cleaner is 70 dB.

ii $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$

$20 = 10\log_{10}\left(\dfrac{I}{I_0}\right)$

$2 = \log_{10}\left(\dfrac{I}{I_0}\right)$

Using the definition of a logarithm:

$10^2 = \dfrac{I}{I_0}$

$100 = \dfrac{I}{I_0}$

$100 I_0 = I$

So the loudness of rustling leaves is 100 times threshold sound.

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DID YOU KNOW?

The origins of logarithms

John Napier (1550–1617), a Scottish theologian and an amateur mathematician, was the first to invent logarithms. These 'natural', or 'Naperian', logarithms were based on e. Napier originally used the compound interest formula to find the value of e.

Napier was also one of the first mathematicians to use decimals rather than fractions. He invented decimal notation, using either a comma or a point. The point was used in England, but some European countries use a comma.

Henry Briggs (1561–1630), an Englishman who was a professor at Oxford, decided that logarithms would be more useful if they were based on 10 (our decimal system). Briggs painstakingly produced a table of common logarithms correct to 14 decimal places.

The work on logarithms was greatly appreciated by **Kepler**, **Galileo** and other astronomers at the time, since they allowed the computation of very large numbers.

Exercise 10.04 Logarithms

1 Evaluate:

a $\log_2 16$ **b** $\log_4 16$ **c** $\log_5 125$

d $\log_3 3$ **e** $\log_7 49$ **f** $\log_7 7$

g $\log_5 1$ **h** $\log_2 128$ **i** $\log_8 8$

2 Evaluate:

a $2^{\log_2 3}$ **b** $7^{\log_7 4}$ **c** $3^{\log_3 29}$

3 Evaluate:

a $3 \log_2 8$ **b** $\log_5 25 + 1$ **c** $3 - \log_3 81$

d $4 \log_3 27$ **e** $2 \log_{10} 10\,000$ **f** $1 + \log_4 64$

g $3 \log_4 64 + 5$ **h** $\dfrac{\log_3 9}{2}$ **i** $\dfrac{\log_8 64 + 4}{\log_2 8}$

4 Evaluate:

a $\log_2 \dfrac{1}{2}$ **b** $\log_3 \sqrt{3}$ **c** $\log_4 2$ **d** $\log_5 \dfrac{1}{25}$

e $\log_7 \sqrt[4]{7}$ **f** $\log_3 \dfrac{1}{\sqrt[3]{3}}$ **g** $\log_4 \dfrac{1}{2}$ **h** $\log_8 2$

i $\log_6 6\sqrt{6}$ **j** $\log_2 \dfrac{\sqrt{2}}{4}$

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5 Evaluate correct to 2 decimal places:

a $\log_{10} 1200$ **b** $\log_{10} 875$ **c** $\log_e 25$

d $\ln 140$ **e** $5 \ln 8$ **f** $\log_{10} 350 + 4.5$

g $\dfrac{\log_{10} 15}{2}$ **h** $\ln 9.8 + \log_{10} 17$ **i** $\dfrac{\log_{10} 30}{\log_e 30}$

6 Write in logarithmic form:

a $3^x = y$ **b** $5^x = z$ **c** $x^2 = y$ **d** $2^b = a$

e $b^3 = d$ **f** $y = 8^x$ **g** $y = 6^x$ **h** $y = e^x$

i $y = a^x$ **j** $Q = e^x$

7 Write in index form:

a $\log_3 5 = x$ **b** $\log_a 7 = x$ **c** $\log_3 a = b$

d $\log_x y = 9$ **e** $\log_a b = y$ **f** $y = \log_2 6$

g $y = \log_3 x$ **h** $y = \log_{10} 9$ **i** $y = \ln 4$

8 Solve for x, correct to 1 decimal place where necessary:

a $\log_{10} x = 6$ **b** $\log_3 x = 5$ **c** $\log_x 343 = 3$

d $\log_x 64 = 6$ **e** $\log_5 \dfrac{1}{5} = x$ **f** $\log_x \sqrt{3} = \dfrac{1}{2}$

g $\ln x = 3.8$ **h** $3 \log_{10} x - 2 = 10$ **i** $\log_4 x = \dfrac{3}{2}$

9 Evaluate y given that $\log_y 125 = 3$.

10 If $\log_{10} x = 1.65$, evaluate x correct to 1 decimal place.

11 Evaluate b to 3 significant figures if $\log_e b = 0.894$.

12 Find the value of $\log_2 1$. What is the value of $\log_a 1$?

13 Evaluate $\log_5 5$. What is the value of $\log_a a$?

14 **a** Evaluate $\ln e$ without a calculator.

b Using a calculator, evaluate:

i $\log_e e^3$ **ii** $\log_e e^2$ **iii** $\ln_e e^5$ **iv** $\log_e \sqrt{e}$

v $\ln_e \dfrac{1}{e}$ **vi** $e^{\ln 2}$ **vii** $e^{\ln 3}$ **viii** $e^{\ln 5}$

ix $e^{\ln 7}$ **x** $e^{\ln 1}$ **xi** $e^{\ln e}$

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15 A class was given musical facts to learn. The students were then tested on these facts and each week they were given similar tests to find out how much they were able to remember. The formula $A = 85 - 55 \log_{10}(t + 2)$ seemed to model the average score after t weeks.

- **a** What was the initial average score?
- **b** What was the average score after:
  - **i** 1 week?
  - **ii** 3 weeks?
- **c** After how many weeks was the average score 30?

16 The pH of a solution is defined as pH = $-\log [H^+]$ where $[H^+]$ is the hydrogen ion concentration. A solution is acidic if its pH is less than 7, alkaline if pH is greater than 7 and neutral if pH is 7. For each question find its pH and state whether it is acidic, alkaline or neutral.

- **a** Fruit juice whose hydrogen ion concentration is 0.0035
- **b** Water with $[H^+] = 10^{-7}$
- **c** Baking soda with $[H^+] = 10^{-9}$
- **d** Coca Cola whose hydrogen ion concentration is 0.01
- **e** Bleach with $[H^+] = 1.2 \times 10^{-12}$
- **f** Coffee with $[H^+] = 0.000\ 01$

17 If $f(x) = \log x$ and $g(x) = 2x - 7$, find:

- **a** $f(g(x))$
- **b** $g(f(x))$

INVESTIGATION

HISTORY OF BASES AND NUMBER SYSTEMS

Common logarithms use base 10 like our decimal number system. We might have developed a different system if we had a different number of fingers! The Mayans, in ancient times, used base 20 for their number system since they counted with both their fingers and toes.

1 Research the history and types of other number systems, including those of Aboriginal and Torres Strait Islander peoples. Did any cultures use systems other than base 10? Why?

2 Explore computer-based systems. Computers have used both binary (base 2) and octal (base 8). Find out why these bases are used.

Logarithm laws

Logarithm laws

10.05 Logarithm laws

Because logarithms are just another way of writing indices (powers), there are logarithm laws that correspond to the index laws.

$$\log_a (xy) = \log_a x + \log_a y$$

Proof

Let $\quad x = a^m \quad$ and $\quad y = a^n$

Then $\quad m = \log_a x$ and $n = \log_a y$

$$\begin{aligned} xy &= a^m \times a^n \\ &= a^{m+n} \end{aligned}$$

$\therefore \quad \log_a (xy) = m + n \qquad$ (by definition)

$\qquad = \log_a x + \log_a y$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Proof

Let $\quad x = a^m \quad$ and $\quad y = a^n$

Then $\quad m = \log_a x \quad$ and $\quad n = \log_a y$

$$\begin{aligned} \frac{x}{y} &= a^m \div a^n \\ &= a^{m-n} \end{aligned}$$

$\therefore \log_a \left(\frac{x}{y}\right) = m - n \qquad$ (by definition)

$\qquad = \log_a x - \log_a y$

$$\log_a x^n = n \log_a x$$

Proof

Let $\quad x = a^m$

Then $\quad m = \log_a x$

$$\begin{aligned} x^n &= (a^m)^n \\ &= a^{mn} \end{aligned}$$

$\therefore \quad \log_a x^n = mn \qquad$ (by definition)

$\qquad = n \log_a x$

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$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$

Proof

$$\begin{aligned}\log_a\left(\frac{1}{x}\right) &= \log_a 1 - \log_a x\\ &= 0 - \log_a x\\ &= -\log_a x\end{aligned}$$

Logarithm laws

EXAMPLE 14

a Given $\log_5 3 = 0.68$ and $\log_5 4 = 0.86$, find:

i $\log_5 12$ **ii** $\log_5 0.75$ **iii** $\log_5 9$ **iv** $\log_5 20$

b Solve $\log_2 12 = \log_2 3 + \log_2 x$.

c Simplify $\log_a 21$ if $\log_a 3 = p$ and $\log_a 7 = q$.

d The formula for measuring R, the strength of an earthquake on the Richter scale, is $R = \log\left(\frac{I}{S}\right)$ where I is the maximum seismograph signal of the earthquake being measured and S is the signal of a standard earthquake.

Show that:

i $\log I = R + \log S$ **ii** $I = S(10^R)$

Solution

a **i**

$$\begin{aligned}\log_5 12 &= \log_5 (3 \times 4)\\ &= \log_5 3 + \log_5 4\\ &= 0.68 + 0.86\\ &= 1.54\end{aligned}$$

ii

$$\begin{aligned}\log_5 0.75 &= \log_5 \frac{3}{4}\\ &= \log_5 3 - \log_5 4\\ &= 0.68 - 0.86\\ &= -0.18\end{aligned}$$

iii

$$\begin{aligned}\log_5 9 &= \log_5 3^2\\ &= 2\log_5 3\\ &= 2 \times 0.68\\ &= 1.36\end{aligned}$$

iv

$$\begin{aligned}\log_5 20 &= \log_5 (5 \times 4)\\ &= \log_5 5 + \log_5 4\\ &= 1 + 0.86\\ &= 1.86\end{aligned}$$

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b $\log_2 12 = \log_2 3 + \log_2 x$

$= \log_2 3x$

So $12 = 3x$

$4 = x$

c $\log_a 21 = \log_a (3 \times 7)$

$= \log_a 3 + \log_a 7$

$= p + q$

d **i** $R = \log\left(\frac{I}{S}\right)$

$= \log I - \log S$

$R + \log S = \log I$

ii $R = \log\left(\frac{I}{S}\right)$

$\frac{I}{S} = 10^R$

$I = S(10^R)$

Change of base

If we need to evaluate logarithms such as $\log_5 2$, we use the change of base formula.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof

Let $y = \log_a x$

Then $x = a^y$

Take logarithms to the base b of both sides of the equation:

$\log_b x = \log_b a^y$

$= y \log_b a$

$\therefore \quad \frac{\log_b x}{\log_b a} = y$

$= \log_a x$

To find the logarithm of any number, such as $\log_5 2$, you can change it to either $\log_{10} x$ or $\log_e x$.

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EXAMPLE 15

a Evaluate $\log_5 2$ correct to 2 decimal places.

b Find the value of $\log_2 3$ to 1 decimal place.

Solution

a $\log_5 2 = \dfrac{\log 2}{\log 5}$

≈ 0.43

b $\log_2 3 = \dfrac{\log 3}{\log 2}$

≈ 1.6

Exercise 10.05 Logarithm laws

1 Simplify:

a $\log_a 4 + \log_a y$

b $\log_a 4 + \log_a 5$

c $\log_a 12 - \log_a 3$

d $\log_a b - \log_a 5$

e $3 \log_x y + \log_x z$

f $2 \log_k 3 + 3 \log_k y$

g $5 \log_a x - 2 \log_a y$

h $\log_a x + \log_a y - \log_a z$

i $\log_{10} a + 4 \log_{10} b + 3 \log_{10} c$

j $3 \log_3 p + \log_3 q - 2 \log_3 r$

k $\log_4 \dfrac{1}{n}$

l $\log_x \dfrac{1}{6}$

2 Evaluate:

a $\log_5 5^2$

b $\log_7 7^6$

3 Given $\log_7 2 = 0.36$ and $\log_7 5 = 0.83$, find:

a $\log_7 10$

b $\log_7 0.4$

c $\log_7 20$

d $\log_7 25$

e $\log_7 8$

f $\log_7 14$

g $\log_7 50$

h $\log_7 35$

i $\log_7 98$

4 Use the logarithm laws to evaluate:

a $\log_5 50 - \log_5 2$

b $\log_2 16 + \log_2 4$

c $\log_4 2 + \log_4 8$

d $\log_5 500 - \log_5 4$

e $\log_9 117 - \log_9 13$

f $\log_8 32 + \log_8 16$

g $3 \log_2 2 + 2 \log_2 4$

h $2 \log_4 6 - (2 \log_4 3 + \log_4 2)$

i $\log_6 4 - 2 \log_6 12$

j $2 \log_3 6 + \log_3 18 - 3 \log_3 2$

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5 If $\log_a 3 = x$ and $\log_a 5 = y$, find an expression in terms of x and y for:

a $\log_a 15$ **b** $\log_a 0.6$ **c** $\log_a 27$

d $\log_a 25$ **e** $\log_a 9$ **f** $\log_a 75$

g $\log_a 3a$ **h** $\log_a \frac{a}{5}$ **i** $\log_a 9a$

6 If $\log_a x = p$ and $\log_a y = q$, find, in terms of p and q:

a $\log_a xy$ **b** $\log_a y^3$ **c** $\log_a \frac{y}{x}$ **d** $\log_a x^2$

e $\log_a xy^5$ **f** $\log_a \frac{x^2}{y}$ **g** $\log_a ax$ **h** $\log_a \frac{a}{y^2}$

i $\log_a a^3y$ **j** $\log_a \frac{x}{ay}$

7 If $\log_a b = 3.4$ and $\log_a c = 4.7$, evaluate:

a $\log_a \frac{c}{b}$ **b** $\log_a bc^2$ **c** $\log_a (bc)^2$

d $\log_a abc$ **e** $\log_a a^2c$ **f** $\log_a b^7$

g $\log_a \frac{a}{c}$ **h** $\log_a a^3$ **i** $\log_a bc^4$

8 Solve:

a $\log_4 12 = \log_4 x + \log_4 3$ **b** $\log_3 4 = \log_3 y - \log_3 7$

c $\log_a 6 = \log_a x - 3 \log_a 2$ **d** $\log_2 81 = 4 \log_2 x$

e $\log_x 54 = \log_x k + 2 \log_x 3$

9 **a** Change the subject of $\text{dB} = 10 \log\left(\frac{I}{I_0}\right)$ to I.

b Find the value of I in terms of I_0 when dB = 45.

10 **a** Show that the formula $A = 100 - 50 \log (t + 1)$ can be written as:

i $\log (t + 1) = \frac{100 - A}{50}$ **ii** $t = 10^{\frac{100-A}{50}} - 1$

b Hence find:

i A when $t = 3$ **ii** t when $A = 75$

11 Evaluate to 2 decimal places:

a $\log_4 9$ **b** $\log_6 25$ **c** $\log_9 200$ **d** $\log_2 12$

e $\log_3 23$ **f** $\log_8 250$ **g** $\log_5 9.5$ **h** $2 \log_4 23.4$

i $7 - \log_7 108$ **j** $3 \log_{11} 340$

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10.06 Logarithmic functions

A **logarithmic function** is a function of the form $y = \log_a x$.

Exponential and log graphs match-up

EXAMPLE 16

Sketch the graph of $y = \log_2 x$.

Solution

y-intercept ($x = 0$): No y-intercept because $x > 0$.

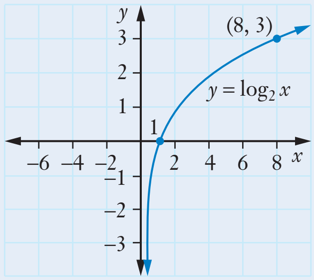
x-intercept ($y = 0$): $0 = \log_2 x$

$x = 2^0 = 1$, so x-intercept is 1 ($y = 0$).

Complete a table of values.

$y = \log_2 x$ means $x = 2^y$. For $x = 6$ in the table, use the change of base formula, $\log_2 x = \dfrac{\log x}{\log 2}$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	6	8
y	−2	−1	0	1	2	2.58	3



Logarithmic functions

- The logarithmic function $y = \log_a x$ is the inverse function of an exponential function $y = a^x$.
- Domain $(0, \infty)$, range $(-\infty, \infty)$.
- $x > 0$ so the curve is always to the right of the y-axis (no y-intercept).
- The y-axis is an **asymptote**.
- The x-intercept is always 1 because $\log_a 1 = 0$.

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EXAMPLE 17

Sketch the graph of:

a $y = \log_e x - 1$ **b** $y = 3\log_{10} x + 4$

Solution

a No y-intercept ($x = 0$) because $\log_e 0$ is undefined. The y-axis is an asymptote.

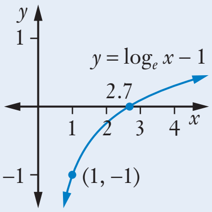
x-intercept ($y = 0$)

$0 = \log_e x - 1$

$1 = \log_e x$

$x = e^1$

≈ 2.7



Notice that this is the graph of $y = \log_e x$ moved down 1 unit.

Complete a table of values for this graph using the ln key on the calculator.

x	1	2	3	4
y	–1	–0.3	0.1	0.4

b Complete a table of values using the log key on the calculator.

$y = 3\log_{10} x + 4$

x	1	2	3	4
y	4	4.9	5.4	5.8

No y-intercept.

For x-intercept, $y = 0$:

$$0 = 3\log_{10} x + 4$$

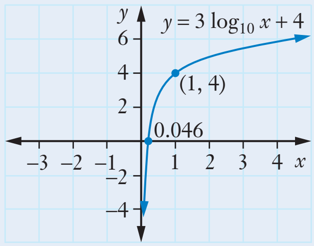
$$-4 = 3\log_{10} x$$

$$-\frac{4}{3} = \log_{10} x$$

$$10^{-\frac{4}{3}} = x$$

$$x = 0.04641\ldots$$

$$\approx 0.046$$



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EXAMPLE 18

a Sketch the graphs of $y = e^x$, $y = \log_e x$ and $y = x$ on the same set of axes.

b What relationship do these graphs have?

c If $f(x) = \log_a x$, sketch the graph of $y = -f(x)$ and state its domain and range.

Solution

a Drawing $y = e^x$ gives an exponential curve with y-intercept 1.

Find another point, say $x = 2$:

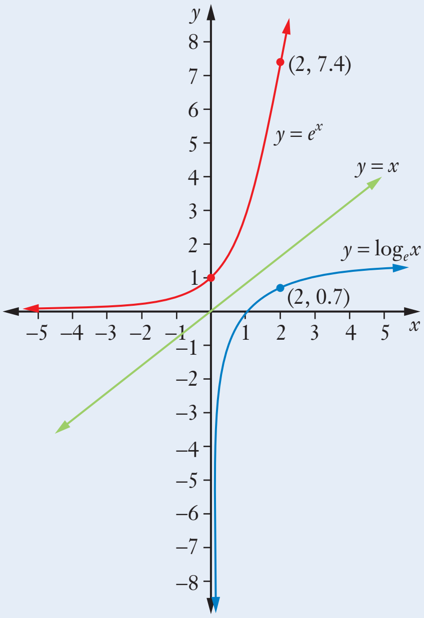
$$\begin{aligned} y &= e^2 \\ &= 7.3890\ldots \\ &\approx 7.4 \end{aligned}$$

Drawing $y = \log_e x$ gives a logarithmic curve with x-intercept 1.

Find another point, say $x = 2$:

$$\begin{aligned} y &= \ln 2 \\ &= 0.6931\ldots \\ &\approx 0.7 \end{aligned}$$

$y = x$ is a linear function with gradient 1 and y-intercept 0.



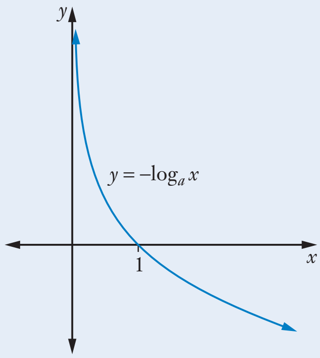
b The graphs of $y = e^x$ and $y = \log_e x$ are reflections of each other in the line $y = x$. They are **inverse functions**.

c Given $f(x) = \log_a x$,

$y = -f(x)$

$= -\log_a x$

This is a reflection of $f(x)$ in the x-axis.



Domain $(0, \infty)$, range $(-\infty, \infty)$

The exponential and logarithmic functions

$f(x) = a^x$ and $f(x) = \log_a x$ are inverse functions. Their graphs are reflections of each other in the line $y = x$.

INVESTIGATION

GRAPHS OF LOGARITHMIC FUNCTIONS

1 Substitute different values of x into the logarithmic function $y = \log x$: positive, negative and zero. What do you notice?

2 Use a graphics calculator or graphing software to sketch the graphs of different logarithmic functions such as

a $y = \log_2 x, y = \log_3 x, y = \log_4 x, y = \log_5 x, y = \log_6 x$

b $y = \log_2 x + 1, y = \log_2 x + 2, y = \log_2 x + 3, y = \log_2 x - 1, y = \log_2 x - 2$

c $y = 2 \log_2 x, y = 3 \log_2 x, y = -\log_2 x, y = -2 \log_2 x, y = -3 \log_2 x$

d $y = 2 \log_2 x + 1, y = 2 \log_2 x + 2, y = 2 \log_2 x + 3, y = 2 \log_2 x - 1, y = 2 \log_2 x - 2$

e $y = 3 \log_4 x + 1, y = 5 \log_3 x + 2, y = -\log_5 x + 3, y = -2 \log_2 x - 1, y = 4 \log_7 x - 2$

3 Try sketching the graph of $y = \log_{-2} x$. What does the table of values look like? Are there any discontinuities on the graph? Why? Could you find the domain and range? Use a graphics calculator or graphing software to sketch this graph. What do you find?

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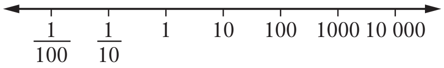
Logarithmic scales

It is difficult to describe and graph exponential functions because their y values increase so quickly. We use logarithms and **logarithmic scales** to solve this problem.

On a base 10 logarithmic scale, an axis or number line has units that don't increase by 1, but by powers of 10.



Examples of base 10 logarithmic scales are:

- the Richter scale for measuring earthquake magnitude
- the pH scale for measuring acidity in chemistry
- the decibel scale for measuring loudness
- the octave (frequency) scale in music

EXAMPLE 19

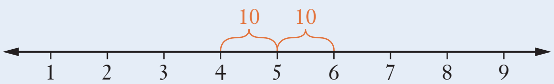
a Ged finds that the pH of soil is 4 in the eastern area of his garden and 6 in the western area. The pH formula is logarithmic and pH < 7 is acidic. What is the difference in acidity in these 2 areas of the garden?

b If Ged finds another area with a pH of 3.6, how much more acidic is this area than the eastern area?

Solution

a The difference in pH between 4 and 6 is 2. But this is a logarithmic scale.

Each interval on a logarithmic scale is a multiple of 10.



So the difference is $10 \times 10 = 10^2 = 100$.

The lower the pH, the more acidic. So the soil in the eastern area is 100 times more acidic than the soil in the western area.

b The difference in pH between 4 and 3.6 is 0.4.

So the difference is $10^{0.4} = 2.5118 \ldots \approx 2.5$.

The soil in this area is about 2.5 times more acidic than the soil in the eastern area.

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Exercise 10.06 Logarithmic functions

1 Sketch the graph of each logarithmic function and state its domain and range.

a $y = \log_3 x$ **b** $f(x) = 2\log_4 x$ **c** $y = \log_2 x + 1$

d $y = \log_5 x - 1$ **e** $f(x) = \log_4 x - 2$ **f** $y = 5\ln x + 3$

g $f(x) = -3\log_{10} x + 2$

2 Sketch the graphs of $y = 10^x$, $y = \log_{10} x$ and $y = x$ on the same number plane. What do you notice about the relationship of the curves to the line?

3 Sketch the graph of $f(x) = \log_2 x$ and $y = \log_2 (-x)$ on the same set of axes and describe their relationship.

4 a Sketch the graphs of $y = \log_2 x$, $y = 2^x$ and $y = x$ on the same set of axes.

b Find the inverse function of $y = \log_2 x$.

5 Find the inverse function of each function.

a $y = \log_7 x$ **b** $y = \log_9 x$ **c** $y = \log_e x$

d $y = 2^x$ **e** $y = 6^x$ **f** $y = e^x$

6 This table lists some of the earthquakes experienced in Australia.

Year	Location	Strength on Richter scale
1989	Newcastle NSW	5.6
1997	Collier Bay WA	6.3
2001	Swan Hill Vic	4.8
2010	Kalgoorlie WA	5.2
2015	Coral Sea Qld	5.5
2017	Orange NSW	4.3
2018	Coffs Harbour NSW	4.2

The Richter scale for earthquakes is logarithmic. Use the table to find the difference in magnitude (correct to the nearest whole number) between the earthquakes in:

a Newcastle and Swan Hill

b Collier Bay and Orange

c Newcastle and Orange

d Coral Sea and Kalgoorlie

e Collier Bay and Coffs Harbour

7 The decibel (dB) scale for loudness is logarithmic. Find (correct to the nearest whole number) the difference in loudness between:

a 20 and 23 dB **b** 40 and 41 dB **c** 65.2 and 66.5 dB

d 85.4 and 88.9 dB **e** 52.3 and 58.6 dB

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10.07 Exponential equations

Exponential equations can be solved using logarithms or the change of base formula.

Exponential equations

WS Logarithmic and exponential equations

WS Solving exponential equations

WS Using exponential models

EXAMPLE 20

Solve $5^x = 7$ correct to 1 decimal place.

Solution

Method 1: Logarithms

Take logarithms of both sides:

$\log 5^x = \log 7$

$x \log 5 = \log 7$

$$x = \frac{\log 7}{\log 5}$$

$= 1.2090\ldots$

≈ 1.2

Method 2: Change of base formula

Convert to logarithm form:

$5^x = 7$ means $\log_5 7 = x$

Using the change of base to evaluate x:

$x = \log_5 7$

$$= \frac{\log 7}{\log 5}$$

$= 1.2090\ldots$

≈ 1.2

EXAMPLE 21

a Solve $e^{3.4x} = 100$ correct to 2 decimal places.

b The temperature T in °C of a metal as it cools down over t minutes is given by $T = 27 + 219e^{-0.032t}$. Find, correct to 1 decimal place, the time it takes to cool down to 100°C.

Exponential functions

Solution

a With an equation involving e we use $\ln x$, which is $\log_e x$.

Take natural logs of both sides:

$\ln e^{3.4x} = \ln 100$

$3.4x = \ln 100$ $\ln x$ and e^x are inverses

$$x = \frac{\ln 100}{3.4}$$

$= 1.3544\ldots$

≈ 1.35

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b When $T = 100$:

$$100 = 27 + 219e^{-0.032t}$$

$$73 = 219e^{-0.032t}$$

$$\frac{73}{219} = e^{-0.032t}$$

$$\log_e\left(\frac{73}{219}\right) = \log_e(e^{-0.032t})$$

$$= -0.032t$$

$$t = \frac{\log_e\left(\frac{73}{219}\right)}{-0.032}$$

$$= 34.3316\ldots$$

$$= 34.3 \text{ to 1 d.p.}$$

So it takes 34.3 minutes to cool down to 100°C.

Exercise 10.07 Exponential equations

1 Solve each equation correct to 2 significant figures:

a $4^x = 9$ **b** $3^x = 5$ **c** $7^x = 14$ **d** $2^x = 15$

e $5^x = 34$ **f** $6^x = 60$ **g** $2^x = 76$ **h** $4^x = 50$

i $3^x = 23$ **j** $9^x = 210$

2 Solve, correct to 2 decimal places:

a $2^x = 6$ **b** $5^y = 15$ **c** $3^x = 20$ **d** $7^m = 32$

e $4^k = 50$ **f** $3^t = 4$ **g** $8^x = 11$ **h** $2^p = 57$

i $4^x = 81.3$ **j** $6^n = 102.6$

3 Solve, to 1 decimal place:

a $3^{x+1} = 8$ **b** $5^{3n} = 71$ **c** $2^{x-3} = 12$

d $4^{2n-1} = 7$ **e** $7^{5x+2} = 11$ **f** $8^{3-n} = 5.7$

g $2^{x+2} = 18.3$ **h** $3^{7k-3} = 32.9$ **i** $9^{\frac{x}{2}} = 50$

4 Solve each equation correct to 3 significant figures:

a $e^x = 200$ **b** $e^{3t} = 5$ **c** $2e^t = 75$

d $45 = e^x$ **e** $3000 = 100e^n$ **f** $100 = 20e^{3t}$

g $2000 = 50e^{0.15t}$ **h** $15\,000 = 2000e^{0.03k}$ **i** $3Q = Qe^{0.02t}$

5 The amount A of money in a bank account after n years grows with compound interest according to the formula $A = 850(1.025)^n$.

a Find:

i the initial amount in the bank **ii** the amount after 7 years.

b Find how many years it will take for the amount in the bank to be \$1000.

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6 The population of a city is given by $P = 35\ 000e^{0.024t}$ where t is time in years.

a Find the population:

i initially **ii** after 10 years **iii** after 50 years.

b Find when the population will reach:

i 80 000 **ii** 200 000

7 A species of wattle is gradually dying out in a Blue Mountains region. The number of wattle trees over time t years is given by $N = 8900e^{-0.048t}$.

a Find the number of wattle trees:

i initially

ii after 5 years

iii after 70 years.

b After how many years will there be:

i 5000 wattle trees?

ii 200 wattle trees?

Photo courtesy Margaret Grove

8 A formula for the mass M g of plutonium after t years is given by $M = 100e^{-0.000\ 03t}$. Find:

a initial mass **b** mass after 50 years **c** mass after 500 years

d its half-life (the time taken to decay to half of its initial mass)

9 The temperature of an electronic sensor is given by the formula $T = 18 + 12e^{0.002t}$ where t is in hours.

a What is the temperature of the sensor after 5 hours?

b When the temperature reaches 50°C the sensor needs to be shut down to cool. After how many hours does this happen?

10 A particle is moving along a straight line with displacement x cm over time t s according to the formula $x = 5e^t + 23$.

a Find:

i the initial displacement

ii the exact velocity after 20 s

iii the displacement after 6 s

iv the time when displacement is 85 cm

v the time when the velocity is 1000 cm s^{-1}.

b Show that acceleration $a = x - 23$.

c Find the acceleration when displacement is 85 cm.

11 EXT1 Find the inverse function of:

a $y = e^{2x}$ **b** $y = \ln(x + 1)$ **c** $f(x) = e^{3x} + 1$

12 EXT1 **a** Find the domain and range of $f(x) = 3^x$.

b What is its inverse function?

c Write down the domain and range of the inverse function.

WS Exponential growth and decay

Exponential decay

EXT1 10.08 Exponential growth and decay

Exponential growth and **exponential decay** are terms that describe a **special rate of change** that occurs in many situations. They describe a quantity that is increasing or decreasing according to an exponential function. Population growth and growth of bacteria in a culture are examples of exponential growth. The decay of radioactive substances and the cooling of a substance are examples of exponential decay.

When a quantity grows or decays exponentially, its rate of change is directly proportional to the current amount of the quantity itself. The more of the quantity there is, the faster it grows or decays. For example, in a population of rabbits, there is fairly slow growth in the numbers at first, but the more rabbits there are, the higher the rate of growth in rabbit numbers.

For exponential growth and decay, the rate of change of a quantity over time is directly proportional to the quantity itself. If we call the quantity N and time t this gives:

$$\frac{dN}{dt} = kN$$

Exponential growth and decay

Given $N = Ae^{kt}$, the rate of exponential growth or decay is $\frac{dN}{dt} = kN$ where k is the **growth or decay constant**.

- For exponential growth, $k > 0$.
- For exponential decay, $k < 0$.
- A is the initial quantity (when $t = 0$) and the N-intercept of the graph.
- The graph of $N = Ae^{kt}$ has a horizontal asymptote on the t-axis.

Proof

$$N = Ae^{kt}$$

$$\begin{aligned}\frac{dN}{dt} &= kAe^{kt}\\ &= kN\end{aligned}$$

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Initially $t = 0$.

$N = Ae^{k \times 0}$

$= A$

$\therefore A$ is the initial quantity.

Sometimes the equation is written as $N = N_0 e^{kt}$.

DID YOU KNOW?

Malthusian Law

Thomas Malthus (1766–1834), at the beginning of the Industrial Revolution, was an economist who developed a theory about population growth that we still use today.

His theory states that under ideal conditions, the birth rate is proportional to the size of the population. That is, $\frac{dN}{dt} = kN$ (Malthusian Law of Population Growth).

Malthus was concerned that the growth rate of populations would be higher than the increase in food supplies, and that people would starve.

Was he right? Is this happening? How could we prove this?

EXAMPLE 22

The population P of a city over t years is given by $P = 20\,000e^{0.04t}$.

a What is the initial population?

b What is the population after 5 years?

c At what rate is the population growing after 5 years?

d Sketch the graph of $P = 20\,000e^{0.04t}$.

Solution

a When $t = 0$:

$P = 20\,000e^0$

$= 20\,000$

So the initial population is 20 000.

b When $t = 5$:

$P = 20\,000e^{0.04 \times 5}$

$= 24\,428.0551\ldots$

$\approx 24\,428$

So the population after 5 years is 24 428.

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c $\dfrac{dP}{dt} = 0.04 \times 20\ 000e^{0.04t}$

$= 800e^{0.04t}$

When $t = 5$:

$\dfrac{dP}{dt} = 800e^{0.04 \times 5}$

$= 977\ 1222\ldots$

≈ 977

So the population is growing by 977 people/year after 5 years.

We could do this another way by using $\dfrac{dN}{dt} = kN$.

$\dfrac{dP}{dt} = kP$

$= 0.04P$

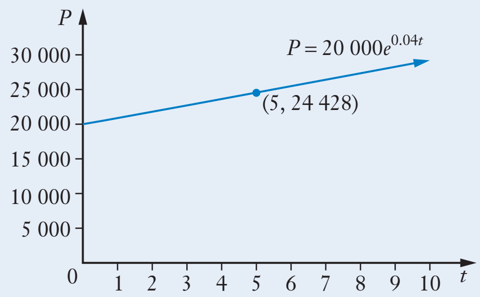
After 5 years, $P = 24\ 428$ (from **b**):

$\dfrac{dP}{dt} = 0.04 \times 24\ 428$

$= 977.12$

≈ 977

d $P = 20\ 000e^{0.04t}$ is an exponential function with P-intercept 20 000. We also know that when $t = 5$, $P = 24\ 428$.



Sometimes you need to find the constants A and k before you can answer questions. When calculating with the exponential function, don't round the decimal value of k. So that your answers are accurate, store the value of k in your calculator's memory or write it down with many decimal places.

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EXAMPLE 23

a The number of bacteria in a culture is given by $N = Ae^{kt}$ where t is measured in hours. If 6000 bacteria increase to 9000 after 8 hours, find:

i k correct to 3 significant figures

ii the number of bacteria after 2 days

iii the rate at which the bacteria are increasing after 2 days

iv when the number of bacteria will reach 1 000 000

b A 50 g mass of uranium decays to 35 g after 2 years. If the rate of decay of its mass is proportional to the mass itself, find the amount of uranium left after 25 years.

Solution

a **i** $N = Ae^{kt}$

When $t = 0$, $N = 6000$

$$6000 = Ae^0$$
$$= A$$

So $N = 6000e^{kt}$

When $t = 8$, $N = 9000$

$$9000 = 6000e^{8k}$$
$$1.5 = e^{8k}$$
$$\log_e 1.5 = \log_e e^{8k}$$
$$= 8k$$
$$\frac{\log_e 1.5}{8} = k$$
$$k = 0.05068\ldots$$
$$\approx 0.0507$$

ii So $N = 6000e^{0.05068\ldots t}$

2 days = 48 hours.

When $t = 48$

$$N = 6000e^{0.05068\ldots \times 48}$$
$$= 68\,344$$

So there will be 68 344 bacteria after 2 days.

iii $\dfrac{dN}{dt} = kN$

$= 0.05068\ldots N$

After 48 hours, $N = 68\ 344$

$\dfrac{dN}{dt} = 0.05068\ldots \times 68\ 344$

$= 3463.8857\ldots$

≈ 3464

or $\dfrac{dN}{dt} = 0.050\ldots(6000e^{0.05068\ldots t})$

$= 304.09\ldots e^{0.05068\ldots t}$

When $t = 48$:

$\dfrac{dN}{dt} = 304.09\ldots e^{0.05068\ldots \times 48}$

$= 3463.8667\ldots$

≈ 3464

So after 2 days the rate of growth will be 3464 bacteria per hour.

Your answer might be slightly different due to rounding of k.

iv When $N = 1\ 000\ 000$

$1\ 000\ 000 = 6000e^{0.05068\ldots t}$

$166.666\ldots = e^{0.05068\ldots t}$

$\log_e 166.6\ldots = \log_e e^{0.05068\ldots t}$

$= 0.05068\ldots t$

$\dfrac{\log_e 166.6\ldots}{0.05068\ldots} = t$

$t = 100.9408\ldots$

≈ 100.9

So the number of bacteria will be 1 000 000 after 100.9 hours.

b $M = Ae^{kt}$ where M is the mass in grams and t is the time in years.

When $t = 0$, $M = 50$

$50 = Ae^0$

$= A$

So $M = 50e^{kt}$

When $t = 2$, $M = 35$

$35 = 50e^{2k}$

$0.7 = e^{2k}$

$\log_e 0.7 = \log_e e^{2k}$

$= 2k$

$\dfrac{\log_e 0.7}{2} = k$

$k = -0.1783\ldots$

So $M = 50e^{-0.1783\ldots t}$

When $t = 25$

$M = 50e^{-0.1783\ldots \times 25}$

$= 0.5790\ldots$

≈ 0.58

So there will be 0.58 grams left after 25 years.

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EXT1 Exercise 10.08 Exponential growth and decay

1 The number of migratory birds in a colony is given by $N = 80e^{0.002t}$, where t is in days.

Photo courtesy Margaret Grove

a How many birds are there in the colony initially?

b How many birds will there be after 30 days?

c After how many days will there be 500 birds?

d Sketch the curve of the population over the first 100 days.

2 The number of bacteria in a culture is given by $N = N_0e^{0.32t}$, where t is time in hours.

a If there are initially 20 000 bacteria, how many will there be after 5 hours?

b How many hours, to the nearest hour, would it take for the number of bacteria to reach 200 000?

3 The rate of decay of radium is proportional to its mass, and 100 kg of radium takes 5 years to decay to 95 kg.

a Show that the mass of radium is given by $M = 100e^{-0.01t}$.

b Find its mass after 10 years.

c Find its half-life (the time taken for the radium to halve its mass).

d Sketch the graph of the decay.

4 A chemical reaction causes the amount of chlorine to be reduced at a rate proportional to the amount of chlorine present at any one time. If the amount of chlorine is given by the formula $A = A_0e^{-kt}$ and 100 L reduces to 65 L after 5 minutes, find:

a the amount of chlorine after 12 minutes

b how long it will take for the chlorine to reduce to 10 L.

5 The production output in a factory increases according to the equation $P = P_0e^{kt}$ where t is in years.

a Find P_0 if the initial output is 5000 units.

b The factory produces 8000 units after 3 years. Find the value of k, to 3 decimal places.

c How many units will the factory produce after 6 years?

d The factory needs to produce 20 000 units to make a maximum profit. After how many years, correct to 1 decimal place, will this happen?

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6 The rate of depletion of rainforests can be estimated as proportional to the area of rainforest. If 3 million m^2 of rainforest is reduced to 2.7 million m^2 after 20 years, find how much rainforest there will be after 50 years.

Photo courtesy Margaret Grove

7 The population of a country is increasing at a yearly rate of 6.9%; that is, $\frac{dP}{dt} = 0.069P$. If the population was 50 000 in 2015, find:

a a formula for the population growth

b the population in the year 2020

c the rate at which the population will be growing in the year 2020

d in which year the population will reach 300 000.

8 An object is cooling according to the formula $T = T_0e^{-kt}$, where T is temperature in degrees Celsius and t is time in minutes. If the temperature is initially 90°C and the object cools down to 81°C after 10 minutes, find:

a its temperature after half an hour

b how long (in hours and minutes) it will take to cool down to 30°C.

9 In the process of the inversion of sugar, the amount of sugar present is given by the formula $S = Ae^{kt}$. If 150 kg of sugar is reduced to 125 kg after 3 hours, find:

a the amount of sugar after 8 hours, to the nearest kilogram

b the rate at which the sugar will be reducing after 8 hours

c how long it will take to reduce to 50 kg.

10 The mass, in grams, of a radioactive substance is given by $M = M_0e^{-kt}$, where t is time in years. Find:

a M_0 and k if a mass of 200 kg decays to 195 kg after 10 years

b the mass after 15 years

c the rate of decay after 15 years

d the half-life of the substance (time taken to decay to half its mass).

11 The number of bacteria in a culture increases from 15 000 to 25 000 in 7 hours. If the rate of bacterial growth is proportional to the number of bacteria at that time, find:

a a formula for the number of bacteria

b the number of bacteria after 12 hours

c how long it will take for the culture to produce 5 000 000 bacteria.

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12 A population in a certain city is growing at a rate proportional to the population itself. After 3 years the population increases by 20%. How long will it take for the population to double?

Photo courtesy Margaret Grove

13 The half-life of radium is 1600 years.

a Find the percentage of radium that will be decayed after 500 years.

b Find the number of years that it will take for 75% of the radium to decay.

14 The population of a city is $P(t)$ at any one time. The rate of decline in population is proportional to the population $P(t)$, that is, $\frac{dP}{dt} = -kP$.

a Show that $P = P_0e^{-kt}$ is a solution of the differential equation $\frac{dP}{dt} = -kP$.

b What percentage decline in population will there be after 10 years, given a 10% decline in 4 years? Answer to the nearest percentage.

c What will the percentage rate of decline in population be after 10 years? Answer to the nearest percentage.

d When will the population fall by 20%? Answer to the nearest 0.1 year.

15 The rate of leakage of water out of a container is proportional to the amount of water in the container at any one time. If the container is 60% empty after 5 minutes, find how long it will take for the container to be 90% empty.

16 Numbers of sheep in a certain district are dropping exponentially due to drought. A survey found that numbers had declined by 15% after 3 years. If the drought continues, how long would it take to halve the number of sheep in that district?

Photo courtesy Margaret Grove

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17 Anthony has a blood alcohol level of 150 mg/dL. The amount of alcohol in the bloodstream decays exponentially. If it decreases by 20% in the first hour, find:

a the level of alcohol in Anthony's blood after 3 hours

b when the blood alcohol level reaches 20 mg/dL.

18 The current C flowing in a conductor dissipates according to the formula $\frac{dC}{dt} = -kC$.

If it dissipates by 40% in 5 seconds, how long will it take to dissipate to 20% of the original current?

19 Pollution levels in a city have been rising exponentially with a 10% increase in pollution levels in the past two years. At this rate, how long will it take for pollution levels to increase by 50%?

20 If $\frac{dQ}{dt} = kQ$, prove that $Q = Ae^{kt}$ satisfies this equation.

INVESTIGATION

EXPONENTIAL DECAY AND THE ENVIRONMENT

Exponential decay is often related to environmental problems such as extinction and lack of sustainability. Choose one or more of the following issues to research.

Photo courtesy Margaret Grove

It is estimated that some animals, such as pandas and koalas, will be extinct soon. How soon will pandas be extinct? Can we do anything to stop this extinction?

How does climate change affect the Earth? Is it affecting us now? If not, how soon will it have a noticeable effect on us?

How long do radioactive substances such as radium and plutonium take to decay? What are some of the issues concerning the storage of radioactive waste?

The erosion and salination of Australian soils are problems that affect our farming. Find out about this issue, and some possible solutions.

The effect of blue-green algae in some of our rivers is becoming a major problem. What steps have been taken to remedy this situation?

Look at the mathematical aspects of these issues. For example, what formulas are used to make predictions? What kinds of time scales are involved in these issues?

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EXT1 10.09 Further exponential growth and decay

WS Further exponential growth and decay

The formulas $\frac{dN}{dt} = kN$ and $N = Ae^{kt}$ are based on the work of **Thomas Malthus** (1766–1834). It is accurate in many cases, but it is a simple formula that doesn't take in the various factors that may influence the rate of population growth. A more realistic formula for exponential growth and decay has the rate of change of a quantity not being directly proportional to the current quantity, but directly proportional to the **difference between the quantity and a constant *P***:

Modified exponential growth and decay

If N has a rate of change proportional to the difference between N and a constant P, then

$$\frac{dN}{dt} = k(N - P).$$

This formula for **modified exponential growth or decay**,

$$N = P + Ae^{kt} \text{ where } A \text{ is a constant,}$$

satisfies the above equation.

$P + A$ is the initial quantity (when $t = 0$) and the N-intercept of the graph.

Its graph is the graph of $N = Ae^{kt}$ shifted up P units with a horizontal asymptote at $N = P$.

Proof

$$N = P + Ae^{kt}$$

$$\begin{aligned}\frac{dN}{dt} &= kAe^{kt}\\ &= k(P + Ae^{kt} - P)\\ &= k(N - P)\end{aligned}$$

The next example shows that some population growths and **Newton's Law of Cooling** both use this formula.

EXAMPLE 24

Further exponential growth

a In a certain town the growth rate in population is given by $\frac{dN}{dt} = k(N - 125)$, where t is measured in years.

- **i** Show that $N = 125 + Ae^{kt}$ is a solution of this equation.
- **ii** If the population is initially 25 650 and after 5 years it is 31 100, find the population after 8 years.
- **iii** After how many years (to 3 significant figures) will the population reach 40 000?

b A saucepan of water is heated to 95°C. The room temperature is a constant 23°C and the water cools down to 55°C after 5 minutes. The cooling rate is proportional to the difference between the water and the room temperatures.

i Show that the equation for the water temperature is $T = 23 + 72e^{-0.162t}$, where T is the temperature in °C and t is the time in minutes.

ii Find the temperature of the water after 30 minutes.

iii When will the water temperature reach 30°C? Give your answer correct to 1 decimal place.

iv Show that the temperature of the water approaches 23°C as t approaches infinity.

Solution

a **i** $N = 125 + Ae^{kt}$

$$\begin{aligned}\frac{dN}{dt} &= kAe^{kt}\\ &= k(125 + Ae^{kt} - 125)\\ &= k(N - 125)\end{aligned}$$

So $N = 125 + Ae^{kt}$ is a solution of the equation.

ii When $t = 0$, $N = 25\ 650$

$$\begin{aligned}25\ 640 &= 125 + Ae^{0}\\ 25\ 525 &= A\end{aligned}$$

So $N = 125 + 25\ 525e^{kt}$

When $t = 5$, $N = 31\ 100$

$$\begin{aligned}31\ 100 &= 125 + 25\ 525e^{5k}\\ 30\ 975 &= 25\ 525e^{5k}\\ 1.2135\ldots &= e^{5k}\\ \ln 1.2135\ldots &= \ln e^{5k}\\ &= 5k\end{aligned}$$

$$\begin{aligned}\frac{\ln 1.2135\ldots}{5} &= k\\ k &= 0.0387\ldots\end{aligned}$$

So $N = 125 + 25\ 525e^{0.0387\ldots t}$

When $t = 8$:

$$\begin{aligned}N &= 125 + 25\ 525e^{0.0387\ldots \times 8}\\ &= 34\ 913.7363\ldots\\ &\approx 34\ 914\end{aligned}$$

So the population after 8 years will be 34 914.

iii When $N = 40\ 000$

$$\begin{aligned}40\ 000 &= 125 + 25\ 525e^{0.0387\ldots t}\\ 39\ 875 &= 25\ 525e^{0.0387\ldots t}\\ 1.5621\ldots &= e^{0.0387\ldots t}\\ \ln 1.5621\ldots &= \ln e^{0.0387\ldots t}\\ &= 0.0387\ldots t\end{aligned}$$

$$\begin{aligned}\frac{\ln 1.5621\ldots}{0.0387\ldots} &= t\\ t &= 11.5255\ldots\\ &\approx 11.5\end{aligned}$$

So the population will be 40 000 after 11.5 years.

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b **i** $T = P + Ae^{kt}$ where $P = 23$ (room temperature).

When $t = 0$, $T = 95$

$95 = 23 + Ae^0$

$72 = A$

So $T = 23 + 72e^k$

When $t = 5$, $T = 55$

$55 = 23 + 72e^{5k}$

$32 = 72e^{5k}$

$0.444\ldots = e^{5k}$

$\ln 0.444\ldots = \ln e^{5k}$

$= 5k$

$\frac{\ln 0.444\ldots}{5} = k$

$k = -0.1621\ldots$

≈ -0.162

So $T = 23 + 72e^{-0.162t}$

ii When $t = 30$

$T = 23 + 72e^{-0.162 \times 30}$

$= 23.5580\ldots$

≈ 23.6

So temperature after 30 minutes will be 23.6°C.

iii When $T = 30$

$30 = 23 + 72e^{-0.162t}$

$7 = 72e^{-0.162t}$

$0.0972\ldots = e^{-0.162t}$

$\ln 0.0972\ldots = \ln e^{-0.162t}$

$= -0.162t$

$\frac{\ln 0.0972\ldots}{-0.162} = t$

$t = 14.3873\ldots$

≈ 14.4

So the temperature will be 30°C after 14.4 minutes.

iv $T = 23 + 72e^{-0.162t}$

$e^{-0.162t} = \frac{1}{e^{0.162t}}$

As $t \to \infty$, $e^{-0.162t} \to 0$

So $72e^{-0.162t} \to 0$

$T \to 23$

$\therefore$ The water's temperature approaches 23°C as t approaches infinity.

For modified exponential decay, $N = P + Ae^{kt}$ and $k < 0$, so as $t \to \infty$, $N \to P$. This means N has a limiting value, whether it be room temperature, carrying capacity or a certain population value.

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EXT1 Exercise 10.09 Further exponential growth and decay

1 **a** Show that $x = 100 + Ae^{2t}$ is a solution of $\frac{dx}{dt} = 2(x - 100)$.

b If $x = 180$ when $t = 3$, find A to 3 significant figures.

c Find t to 3 significant figures when $x = 150$.

2 **a** Show that $N = 45 + Ae^{0.14t}$ is a solution of $\frac{dN}{dt} = 0.14(N - 45)$.

b Given $N = 82$ when $t = 2$, find A to 2 decimal places.

c What is N when $t = 5$?

d Find t when $N = 120$.

e Sketch the graph of this function for values of t from 0 to 25.

3 The rate of change in the volume of water in a dam is given by $\frac{dV}{dt} = k(V - 5000)$, where k is a constant.

a Show that a solution of this differential equation is $V = 5000 + Ae^{kt}$.

b If the initial volume is 87 000 kL and after 10 hours the volume is 129 000 kL, find the values of A and k.

c What volume of water will be in the dam after 3 days?

d Calculate how long it will take for the volume to reach 4.2 million kL. Answer in days and hours, to the nearest hour.

4 **a** Show that $N = P + Ae^{kt}$ is a solution of $\frac{dN}{dt} = k(N - P)$ where k, P and A are constants.

b $\frac{dN}{dt} = k(N - 1000)$ and initially $N = 1500$. When $t = 5$, $N = 2200$. Find t when $N = 2500$.

5 According to Newton's Law of Cooling, the rate of change in the temperature of an object is proportional to the difference between its temperature and the temperature of the air (or surrounding matter). The air temperature is assumed to be constant.
A piece of metal is heated to 80°C and placed in a room where the temperature is a constant 18°C.

a If the metal cools to 68°C after 15 minutes, show that $T = 18 + 62e^{-0.0143t}$.

b When will the temperature reach 30°C?

c Show that as t approaches infinity, the temperature of the metal approaches room temperature.

6 The population of an ant colony is given by $P = 950 + Ae^{kt}$. If there are initially 14 000 ants and after 6 weeks there are 20 000, find:

a the ant population after 10 weeks

b when the population will reach 1 million.

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7 A piece of meat, initially at 14°C, is placed in a freezer whose temperature is a constant –10°C. After 25 seconds the temperature of the meat is 11°C. Find:

a the meat's temperature after 5 minutes

b when the temperature will reach –8°C (to the nearest minute).

8 When a body falls, the rate of change in velocity is given by $\frac{dv}{dt} = -k(v - P)$, where k and P are constants.

a Show that $v = P + Ae^{-kt}$ is a solution of this differential equation.

b When $P = 500$, initial velocity is 0 and velocity v after 5 seconds is 21 m s^{-1}. Find values of A and k.

c Find the velocity after 20 seconds.

d Find the maximum possible velocity as t tends to infinity.

9 An ice-block with temperature –14°C is left out in the sun. The air temperature is a constant 25°C and after 40 seconds the temperature of the ice-block has reached –5°C. Find:

a its predicted temperature after 2 minutes

b when the ice-block will start to melt (i.e. when its temperature will reach 0°C).

10 The population of sheep on a farm is given by $\frac{dN}{dt} = k(N - 1800)$, where N is the number of sheep. If there are initially 3000 sheep and after 3 years there are 3400 sheep, find:

a the number of sheep on the farm after 5 years

b when the sheep population will reach 8000.

Photo courtesy Margaret Grove

11 Wilhemy's Law states that the rate of transformation of a substance in a chemical reaction is proportional to its concentration. That is, $\frac{dx}{dt} = k(x - c)$, where x is the amount of substance transformed and c is the initial concentration of the substance. Initially none of the substance is transformed. If the initial concentration is 7.9 and the amount transformed after 2 minutes is 2.7, find how much of the substance will be transformed after 5 minutes.

12 The rate of growth of a certain town's population is proportional to the excess of the population over 10 000. If the town initially has 18 000 people and after 4 years the population grows to 25 000, find how long it would take to:

a reach 40 000

b be double the initial population.

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13 A formula for the rate of change in population of a certain species of animal is given by $P = 200 + 1600e^{-kt}$. If the population reduces to half after 56 years, find how long it would take to reduce to a quarter of the original population.

14 A saucepan of water is brought nearly to the boil then removed from the heat. The temperature reaches 95°C and room temperature is a constant 25°C. If the water has lost 25% of its excess heat after 2.5 minutes, find how long it will take for it to cool down to 30°C.

15 The velocity of a particle is given by $v = 100 + 280e^{-kt}$. If the velocity decreases by 20% after 50 seconds, find the percentage decrease in velocity after 3 minutes.

16 A population of herons is increasing according to the formula $P = 800 + 2000e^{kt}$ where t is measured in years.

a What is the initial heron population?

b If the population has increased by 20% in 5 years, how long will it take to double the population?

c If the carrying capacity of this region is 7500 herons, how long it will take to reach this capacity?

Photo courtesy Margaret Grove

17 The ozone layer over a certain region is decreasing according to the equation $Q = 50 + 80e^{-kt}$ where t is measured in years. If it decreases by 4% over 10 years, find:

a by what percentage it decreases over 50 years

b how long it takes to decrease by 40%, to the nearest year.

18 The production of a mine is decreasing exponentially, and in the past 5 years there has been a decline of 18%. If production declines by 90%, the mine will close. The equation of production P after t years is given by $P = 500 + 6500e^{-kt}$. Find:

a the percentage of production decline after 10 years

b how long it will take for the mine to close.

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10. TEST YOURSELF

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 Simplify $\log_a 15 - \log_a 3$:

A $\log_a 45$ **B** $\dfrac{\log_a 15}{\log_a 3}$ **C** $\log_a 15 \times \log_a 3$ **D** $\log_a 5$

2 Write $a^x = y$ as a logarithm.

A $\log_y x = a$ **B** $\log_a y = x$ **C** $\log_a x = y$ **D** $\log_x a = y$

3 Solve $5^x = 4$ (there is more than one answer).

A $x = \dfrac{\log 4}{\log 5}$ **B** $x = \dfrac{\log 5}{\log 4}$ **C** $x = \dfrac{\ln 4}{\ln 5}$ **D** $x = \dfrac{\ln 5}{\ln 4}$

4 Evaluate:

a $\log_2 8$ **b** $\log_7 7$ **c** $\log_{10} 1000$ **d** $\log_9 81$

e $\log_e e$ **f** $\log_4 64$ **g** $\log_9 3$ **h** $\log_2 \dfrac{1}{2}$

i $\log_5 \dfrac{1}{25}$ **j** $\ln e^3$

5 Evaluate to 3 significant figures:

a $e^2 - 1$ **b** $\log_{10} 95$ **c** $\log_e 26$ **d** $\log_4 7$

e $\log_4 3$ **f** $\ln 50$ **g** $e + 3$ **h** $\dfrac{5e^3}{\ln 4}$

6 Evaluate:

a $e^{\ln 6}$ **b** $e^{\ln 2}$

7 Write in index form:

a $\log_3 a = x$ **b** $\ln b = y$ **c** $\log c = z$

8 If $\log_7 2 = 0.36$ and $\log_7 3 = 0.56$, find the value of:

a $\log_7 6$ **b** $\log_7 8$ **c** $\log_7 1.5$

d $\log_7 14$ **e** $\log_7 3.5$

9 Solve:

a $3^x = 8$ **b** $2^{3x-4} = 3$ **c** $\log_x 81 = 4$ **d** $\log_6 x = 2$

10 Solve $12 = 10e^{0.01t}$.

11 Evaluate $\log_9 8$ to 1 decimal place.

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12 EXT1 A radioactive substance decays by 10% after 80 years.

- **a** By how much will it decay after 500 years?
- **b** When will it decay to a quarter of its mass?

13 Simplify:

- **a** $5 \log_a x + 3 \log_a y$
- **b** $2 \log_x k - \log_x 3 + \log_x p$

14 Evaluate to 2 significant figures:

- **a** $\log_{10} 4.5$
- **b** $\ln 3.7$

15 Sketch the graph of $y = 2^x + 1$ and state its domain and range.

16 Solve:

- **a** $2^x = 9$
- **b** $3^x = 7$
- **c** $5^{x+1} = 6$
- **d** $4^{2y} = 11$
- **e** $8^{3n-2} = 5$
- **f** $\log_x 16 = 4$
- **g** $\log_3 y = 3$
- **h** $\log_7 n = 2$
- **i** $\log_x 64 = \frac{1}{2}$
- **j** $\log_8 m = \frac{1}{3}$

17 Write as a logarithm:

- **a** $2^x = y$
- **b** $5^a = b$
- **c** $10^x = y$
- **d** $e^x = z$
- **e** $3^{x+1} = y$

18 EXT1 A bird population of 8500 increases to 12 000 after 5 years. Find:

- **a** the population after 10 years
- **b** the rate at which the population is increasing after 10 years
- **c** when the population reaches its carrying capacity of 30 000.

19 Sketch the graph of:

- **a** $y = 5(3^{x+2})$
- **b** $y = 2(3^x) - 5$
- **c** $f(x) = -3^x$
- **d** $y = 3(2^{-x})$

20 Sketch the graph of:

- **a** $f(x) = \log_3 x$
- **b** $y = 3 \ln x - 4$

21 If $\log_x 2 = a$ and $\log_x 3 = b$ find in terms of a and b:

- **a** $\log_x 6$
- **b** $\log_x 1.5$
- **c** $\log_x 8$
- **d** $\log_x 18$
- **e** $\log_x 27$

22 The formula for loudness is $L = 10 \log\left(\frac{I}{I_0}\right)$ where I_0 is threshold sound and L is measured in decibels (dB). Find:

- **a** the dB level of a $5500I_0$ sound
- **b** the sound in terms of I_0 if its dB level is 32.

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23 Simplify:

a $\log_a \frac{1}{x}$ **b** $\log_e \frac{1}{y}$

24 Evaluate:

a $\log_6 12 + \log_6 3$ **b** $\log 25 + \log 4$ **c** $2 \log_4 8$

d $\log_8 72 - \log_8 9$ **e** $\log 53\,000 - \log 53$

25 EXT1 A city doubles its population in 25 years. If it is growing exponentially, when will it triple its population?

26 Solve correct to 1 decimal place:

a $e^x = 15$ **b** $2.7^x = 21.8$ **c** $10^x = 128.7$

27 The amount of money in the bank after n years is given by $A = 5280(1.019)^n$.

a Find the amount in the bank:

i initially **ii** after 3 years **iii** after 4 years.

b Find how long it will take for the amount of money in the bank to reach:

i \$6000 **ii** \$10 000

28 Differentiate each function.

a $y = e^{3x}$ **b** $y = e^{-2x}$ **c** $y = 5e^{4x}$

d $y = -2e^{8x} + 5x^3 - 1$ **e** $y = x^2e^{2x}$ **f** $y = (4e^{3x} - 1)^9$

g $y = \frac{x}{e^{2x}}$

29 The formula for the number of wombats in a region of New South Wales after t years is $N = 1118 - 37e^{0.032t}$.

a Find the initial number of wombats in this region.

b How many wombats are there after 5 years?

c How long will it take until the number of wombats in the region is:

i 500? **ii** 100?

30 Differentiate:

a $y = e^x + x$ **b** $y = -4e^x$ **c** $y = 3e^{-x}$

d $y = (3 + e^x)^9$ **e** $y = 3x^5e^x$ **f** $y = \frac{e^x}{7x - 2}$

31 An earthquake has magnitude 6.7 and its aftershock has magnitude 4.7 on the base 10 logarithmic Richter scale. How much larger is the first earthquake?

32 Shampoo A has pH 7.2 and shampoo B has pH 8.5. The pH scale is base 10 logarithmic. How much more alkaline is shampoo B?

33 If $f(x) = \log_e x$, $g(x) = e^x$ and $h(x) = 6x^2 - 1$, find:

a $f(h(x))$ **b** $g(h(x))$ **c** $h(g(x))$

d $f(g(x))$ **e** $g(f(x))$ **f** EXT1 $f^{-1}(x)$

34 EXT1 The rate of change in temperature T of a metal over time t minutes as it cools is given by the differential equation $\frac{dT}{dt} = -k(T - 25)$. The metal is initially at 320° and cools to 285° after 3 minutes.

a Show that $T = 25 + Ae^{-kt}$ is a solution of the differential equation.

b Evaluate A and k.

c Find the temperature after half an hour.

d When will the temperature drop to 30°, to the nearest minute?

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10. CHALLENGE EXERCISE

1 If $\log_b 2 = 0.6$ and $\log_b 3 = 1.1$, find:

a $\log_b 6b$ **b** $\log_b 8b$ **c** $\log_b 1.5b^2$

2 Find the point of intersection of the curves $y = \log_e x$ and $y = \log_{10} x$.

3 EXT1 The population of a flock of birds over t years is given by the formula $P = P_0 e^{0.0151t}$.

a How long will it take, correct to 1 decimal place, to increase the population by 35%?

b What will be the percentage increase in population after 10 years, to the nearest per cent?

4 Sketch the graph of $y = \log_2 (x - 1)$ and state its domain and range.

5 By substituting $u = 3^x$, solve $3^{2x} - 3^x - 2 = 0$ correct to 2 decimal places.

6 The pH of a solution is given by $\text{pH} = -\log [\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration.

a Show that pH could be given by $\text{pH} = \log \dfrac{1}{[\text{H}^+]}$.

b Show that $[\text{H}^+] = \dfrac{1}{10^{\text{pH}}}$.

c Find the hydrogen ion concentration, to 1 significant figure, of a substance with a pH of:

i 6.3 **ii** 7.7

7 If $y = 8 + \log_2 (x + 2)$:

a show that $x = 2(2^{y-9} - 1)$

b find, correct to 2 decimal places:

i y when $x = 5$ **ii** x when $y = 1$

8 Find the equation of **a** the tangent and **b** the normal to the curve $y = 3e^x - 5$ at the point $(2, 3e^2 - 5)$.

9 EXT1 Given $f(x) = e^x$, sketch the graph of:

a $y = \dfrac{1}{f(x)}$ **b** $y^2 = f(x)$

10 EXT1 The Logistic Law of Population Growth, first proposed by Belgian mathematician Pierre Verhulst in 1837, is given by $\dfrac{dN}{dt} = kN - bN^2$, where k and b are constants.

Show that the equation $N = \dfrac{kN_0}{bN_0 + (k - bN_0)e^{-kt}}$ is a solution of this differential equation (N_0 is a constant).

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