4. FUNCTIONS

Functions and their graphs are used in many areas such as mathematics, science and economics. In this chapter you will explore what functions are and how to sketch some types of graphs, including straight lines, parabolas and cubics.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- understand the definition of a function and use function notation
- test a function using the vertical line test
- identify a one-to-one function using the horizontal line test
- find the domain and range of functions including composite functions using interval notation
- identify even and odd functions
- understand a linear function, its graph and properties, including the gradient and axes intercepts
- graph situations involving direct linear variation
- find the equation of a line, including parallel and perpendicular lines
- identify a quadratic function, its graph and properties, including its axis of symmetry, turning point and axes intercepts
- solve quadratic equations and use the discriminant to identify the numbers and types of solutions
- find the quadratic equation of a parabola
- EXT1 solve quadratic inequalities
- identify a cubic function, its graph and properties, including the shape, horizontal point of inflection and axes intercepts
- find a cubic equation
- identify a polynomial and its characteristics
- draw the graph of a polynomial showing intercepts
- solve simultaneous equations involving linear and quadratic equations, both algebraically and graphically, and solve problems involving intersection of graphs of functions (for example, break-even points)

TERMINOLOGY

angle of inclination The angle a straight line makes with the positive x-axis measured anticlockwise

axis of symmetry A line that divides a shape into halves that are mirror-images of each other

break-even point The point at which a business' income equals its costs, making neither a profit nor a loss

coefficient A constant multiplied by a pronumeral in an algebraic term. For example, in ax^3 the a is the coefficient

constant term The term in a polynomial function that is independent of x

cubic function A function with x^3 as its highest power or degree

degree The highest power of x in a polynomial

dependent variable A variable whose value depends on another (independent) variable, such as y (depending on x)

direct variation A relationship between two variables such that as one variable increases so does the other, or as one variable decreases so does the other. One variable is a multiple of the other, with equation $y = kx$. Also called **direct proportion**

discriminant The expression $b^2 - 4ac$ that shows how many roots the quadratic equation $ax^2 + bx + c = 0$ has

domain The set of all possible values of x for a function or relation; the set of 'input' values

even function A function $f(x)$ that has the property $f(-x) = f(x)$; its graph is symmetrical about the y-axis

function A relation where every x value in the domain has a unique y value in the range

gradient The steepness of a graph at a point on the graph, measured by the ratio $\frac{\text{rise}}{\text{run}}$; or the change in y values as x values change

horizontal line test A test that checks if a function is one-to-one, whereby any horizontal line drawn on the graph of a function should cut the graph at most once. If the horizontal line cuts the graph more than once, it is not one-to-one

independent variable A variable whose value does not depend on another variable; for example, x in $y = f(x)$

intercepts The values where a graph cuts the x- and y- axes

interval notation A notation that represents an interval by writing its endpoints in square brackets [ ] when they are included and in parentheses ( ) when they are not included

leading coefficient The coefficient of the highest power of x. For example, $2x^4 - x^3 + 3x + 1$ has a leading coefficient of 2

leading term The term with the highest power of x. For example, $2x^4 - x^3 + 3x + 1$ has a leading term of $2x^4$

linear function A function with x as its highest power or degree

monic polynomial A polynomial whose leading coefficient is 1

odd function A function $f(x)$ that has the property $f(-x) = -f(x)$; its graph has point symmetry about the origin (0, 0)

one-to-one function A function in which every y value in the range corresponds to exactly one x value in the domain

parabola The graph of a quadratic function

piecewise function A function that has different functions defined on different intervals

point of inflection A point on a curve where the concavity changes, such as the turning point on the graph of a cubic function

polynomial An expression in the form $P(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$ where n is a positive integer or zero

quadratic function A function with x^2 as the highest power of x

range The set of all possible y values of a function or relation; the set of 'output' values

root A solution of an equation

turning point Where a graph changes from increasing to decreasing or vice versa; sometimes a turning point (horizontal inflection) where concavity changes

vertex A turning point

vertical line test A test that checks if a relation is a function, whereby any vertical line drawn on the graph of a relation should cut the graph at most once. If the vertical line cuts the graph more than once, it is not a function

zero An x value of a function or polynomial for which the y value is zero, that is, $f(x) = 0$

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4.01 Functions

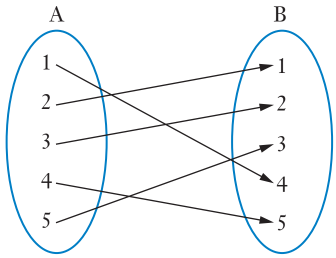
WS
Functions and relations

A **relation** is a set of **ordered pairs** (x, y) where the **variables** x and y are related according to some pattern or rule. The x is called the **independent variable** and the y is called the **dependent variable** because the value of y depends on the value of x. We usually choose a value of x and use it to find the corresponding value of y.

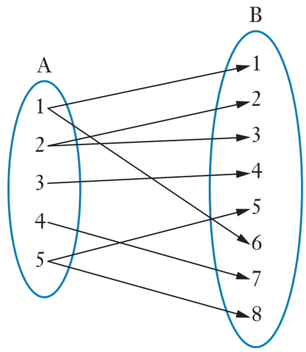
A relation can also be described as a mapping between 2 sets of numbers, with the set of x values, A, on the left and the set of y values B, on the right.

Types of relations

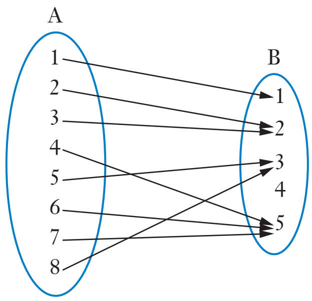
A **one-to-one** relation is a mapping where every element of A corresponds with exactly one element of B and every element of B corresponds with exactly one element of A. Each element has its own unique match.



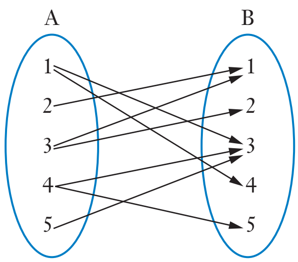
A **one-to-many** relation is a mapping where an element of A corresponds with 2 or more elements of B. For example, 5 in set A matches with 5 and 8 in set B.



A **many-to-one** relation is a mapping where 2 or more elements of A correspond with the same one element of B. For example, 4, 6 and 7 in set A match with 5 in set B.



A **many-to-many** relation is a mapping where 2 or more elements of A correspond with 2 or more elements of B. This is a combination of the one-to-many and many-to-one relations.



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Function

A **function** is a special type of relation where for every value of x there is a unique value of y.

The **domain** is the set of all values of x for which a function is defined.

The **range** is the set of all values of y as x varies.

A function could be a one-to-one or many-to-one relation.

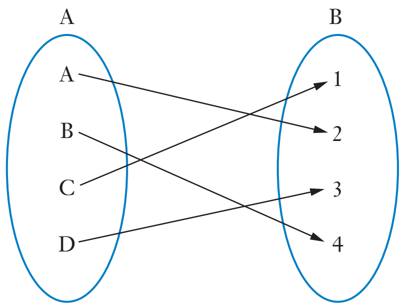
For example, this table matches a group of people with their eye colours.

Person	Anne	Jacquie	Donna	Hien	Marco	Russell	Trang
Colour	Blue	Brown	Grey	Brown	Green	Brown	Brown

The ordered pairs are (Anne, Blue), (Jacquie, Brown), (Donna, Grey), (Hien, Brown), (Marco, Green), (Russell, Brown) and (Trang, Brown).

This table represents a function, since for every person there is a unique eye colour. The domain is the set of people; the range is the set of eye colours. It is a many-to-one function since more than one person can correspond to one eye colour.

Here is a different function:



Set A is the domain, set B is the range.

The ordered pairs are (A, 2), (B, 4), (C, 1) and (D, 3).

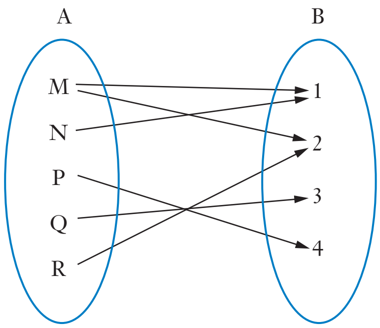
It is a function because every x value in A corresponds to exactly one x value in B.

It is a one-to-one function because every y value in B corresponds to exactly one x value in A.

Here is an example of a relation that is **not** a function. Can you see why?

In this example the ordered pairs are (M, 1), (M, 2), (N, 1), (P, 4), (Q, 3) and (R, 2).

Notice that M corresponds to 2 values in set B: 1 and 2. This means that it is **not** a function. Notice also that M and R both correspond with the same value 2. This is a many-to-many relation.



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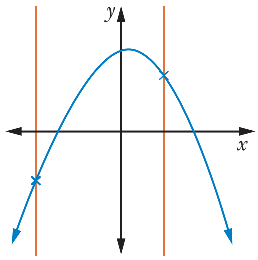
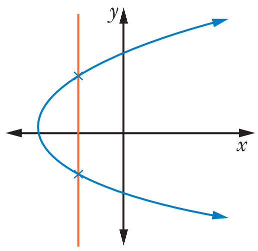
The vertical line test

Relations can also be described by algebraic rules or equations such as $y = x^2 + 1$ and $x^2 + y^2 = 4$, and hence graphed on a number plane. There is a very simple test called the **vertical line test** to test if a graph represents a function.

If any vertical line crosses a graph at only one point, the graph represents a function. This shows that, for every value of x, there is only one value of y.



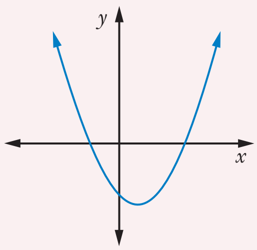
If any vertical line crosses a graph at more than one point, the graph does not represent a function. This shows that, for some value of x, there is more than one value of y.



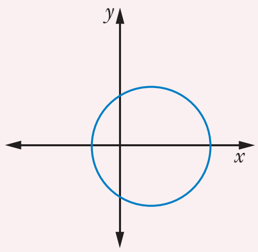
EXAMPLE 1

Does each graph or set of ordered pairs represent a function?

a

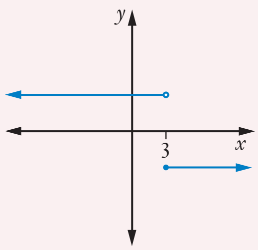


b



c (–2, 3), (–1, 4), (0, 5), (1, 3), (2, 4)

d



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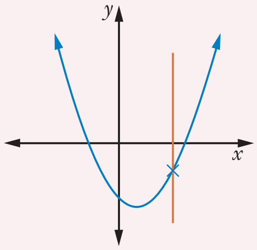
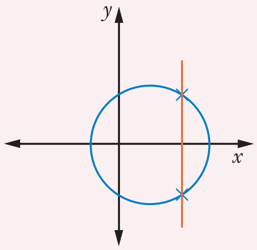
Solution

a



A vertical line only cuts the graph once. So the graph represents a function.

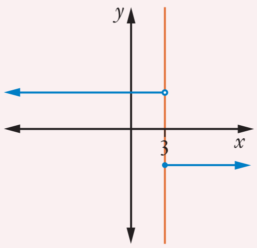
b



A vertical line can cut the curve in more than one place. So the circle does not represent a function.

c For each x value there is only one y value, so this set of ordered pairs is a function.

d



The open circle at $x = 3$ on the top line means that $x = 3$ is not included, while the closed circle on the bottom line means that $x = 3$ is included on this line.

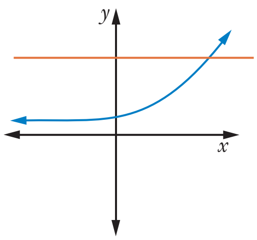
So a vertical line only touches the graph once at $x = 3$.

The graph represents a function.

The horizontal line test

The **horizontal line test** is used on the graph of a function to test whether the function is **one-to-one**.

If any horizontal line crosses a graph at only one point, there is only one x value for every y value. The graph represents a one-to-one function.



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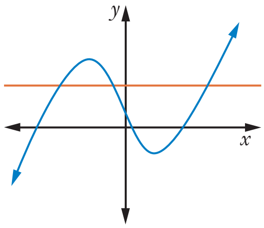
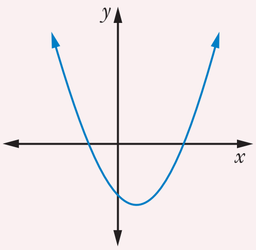
If any horizontal line crosses a graph at more than one point, this means that there are 2 or more x values that have the same y value. The graph does not represent a one-to-one function.



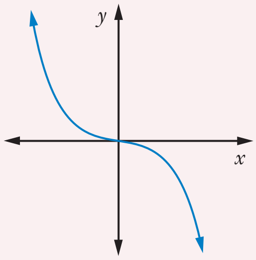
EXAMPLE 2

Does each graph represent a one-to-one function?

a

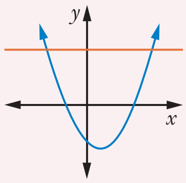


b

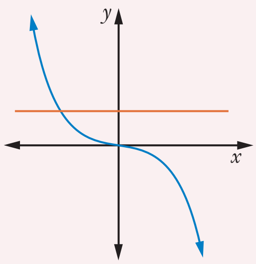


Solution

a A horizontal line cuts the curve in more than one place. The function is not one-to-one.



b A horizontal line cuts the curve in only one place. The function is one-to-one.



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DID YOU KNOW?

René Descartes

The number plane is called the **Cartesian plane** after René Descartes (1596–1650). Descartes used the number plane to develop analytical geometry. He discovered that any equation with two unknown variables can be represented by a line. The points in the number plane can be called Cartesian coordinates.

Descartes used letters at the beginning of the alphabet to stand for numbers that are known, and letters near the end of the alphabet for unknown numbers. This is why we still use x and y so often!

Research Descartes to find out more about his life and work.

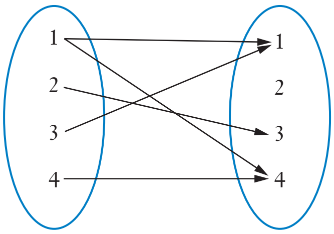
Exercise 4.01 Functions

1 List the ordered pairs for each relation, then state whether the relation is a one-to-one, one-to-many, many-to-one or many-to-many.

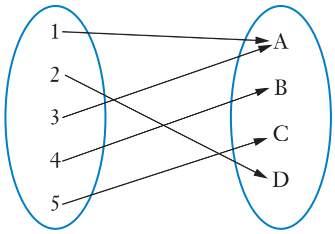
a

Name	Wade	Scott	Geoff	Deng	Mila	Stevie
Hair colour	Black	Blond	Grey	Black	Brown	Blond

b



c



d

x	3	5	8	9	5	8
y	5	± 2	-7	3	6	0

e

x	y
1	9
2	15
3	27
4	33
5	45

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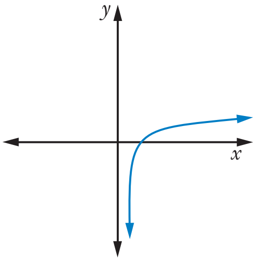
2 Does each graph or set of ordered pairs represent a function? If it does, state whether it is one-to-one.

a



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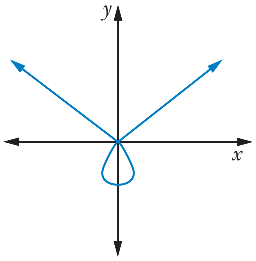
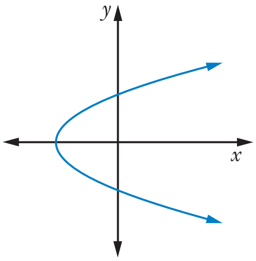
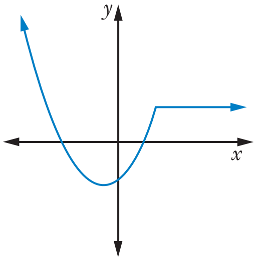


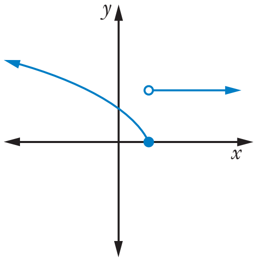
c



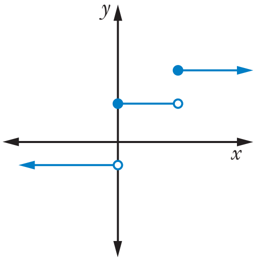
d



e



f



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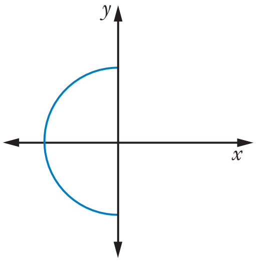

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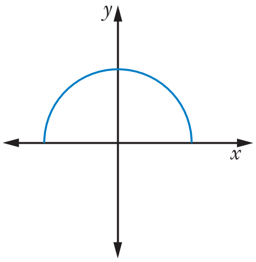
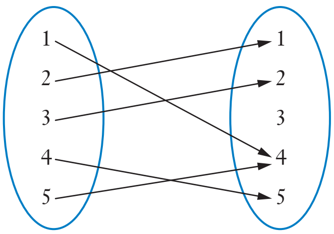


i (1, 3), (2, –1), (3, 3), (4, 0)

j (1, 3), (2, –1), (2, 7), (4, 0)

k



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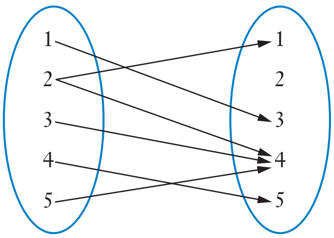

m (2, 5), (3, –1), (4, 0), (–1, 3), (–2, 7)

n

Person	Ben	Paula	Pierre	Hamish	Jacob	Leanne	Pierre	Lien
Sport	Tennis	Football	Tennis	Football	Football	Badminton	Football	Badminton

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o

A	3
B	4
C	7
D	3
E	5
F	7
G	4

3 A relation consists of the ordered pairs (–3, 4), (–1, 5), (0, –2), (1, 4) and (6, 8).

a Write the set of independent variables, x.

b Write the set of dependent variables, y.

c Describe the relation as one-to-one, one-to-many, many-to-one or many-to-many.

d Is the relation a function?

WS

Function notation

4.02 Function notation

Since the value of y depends on the value of x, we say that y is a function of x. We write this using **function notation** as $y = f(x)$.

EXAMPLE 3

a Find the value of y when $x = 3$ in the equation $y = 2x - 1$.

b Evaluate $f(3)$, given $f(x) = 2x - 1$.

Solution

a When $x = 3$:

$$\begin{aligned} y &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

b

$$\begin{aligned} f(x) &= 2x - 1 \\ f(3) &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

Both questions in Example 3 are the same, but the second one looks different because it uses function notation.

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EXAMPLE 4

a If $f(x) = x^2 + 3x + 1$, find $f(-2)$.

b If $f(x) = x^3 - x^2$, find the value of $f(-1)$.

c Find the values of x for which $f(x) = 0$ given that $f(x) = x^2 + 3x - 10$.

Solution

a
$$\begin{aligned} f(x) &= x^2 + 3x + 1 \\ f(-2) &= (-2)^2 + 3(-2) + 1 \\ &= 4 - 6 + 1 \\ &= -1 \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^3 - x^2 \\ f(-1) &= (-1)^3 - (-1)^2 \\ &= -1 - 1 \\ &= -2 \end{aligned}$$

c
$$\begin{aligned} f(x) &= 0 \\ x^2 + 3x - 10 &= 0 \\ (x + 5)(x - 2) &= 0 \end{aligned}$$
$x = -5, x = 2$

A **piecewise function** is a function made up of 2 or more functions defined on different intervals.

EXAMPLE 5

a
$$f(x) = \begin{cases} 3x + 4 & \text{when } x \geq 2 \\ -2x & \text{when } x < 2 \end{cases}$$

Find $f(3), f(2), f(0)$ and $f(-4)$.

b
$$g(x) = \begin{cases} x^2 & \text{when } x > 2 \\ 2x - 1 & \text{when } -1 \leq x \leq 2 \\ 5 & \text{when } x < -1 \end{cases}$$

Find $g(1) + g(-2) - g(3)$.

Solution

a
$$\begin{aligned} f(3) &= 3(3) + 4 && \text{since } 3 \geq 2 \\ &= 13 \\ f(2) &= 3(2) + 4 && \text{since } 2 \geq 2 \\ &= 10 \\ f(0) &= -2(0) && \text{since } 0 < 2 \\ &= 0 \\ f(-4) &= -2(-4) && \text{since } -4 < 2 \\ &= 8 \end{aligned}$$

b
$$\begin{aligned} g(1) &= 2(1) - 1 && \text{since } -1 \leq 1 \leq 2 \\ &= 1 \\ g(-2) &= 5 && \text{since } -2 < -1 \\ g(3) &= 3^2 && \text{since } 3 > 2 \\ &= 9 \end{aligned}$$

So $g(1) + g(-2) - g(3) = 1 + 5 - 9$
$= -3$

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You can also substitute pronumerals instead of numbers into functions.

EXAMPLE 6

Find $f(h+1)$ given $f(x) = 5x + 4$.

Solution

Substitute $h + 1$ for x:

$$\begin{aligned} f(h+1) &= 5(h+1) + 4 \\ &= 5h + 5 + 4 \\ &= 5h + 9 \end{aligned}$$

DID YOU KNOW?

Leonhard Euler

Leonhard Euler (1707–83), from Switzerland, studied functions and invented the function notation $f(x)$. He studied theology, astronomy, medicine, physics and oriental languages as well as mathematics, and wrote more than 500 books and articles on mathematics. He found time between books to marry and have 13 children, and even when he went blind he kept on having books published.

Exercise 4.02 Function notation

1 Given $f(x) = x + 3$, find $f(1)$ and $f(-3)$.

2 If $h(x) = x^2 - 2$, find $h(0)$, $h(2)$ and $h(-4)$.

3 If $f(x) = -x^2$, find $f(5)$, $f(-1)$, $f(3)$ and $f(-2)$.

4 Find the value of $f(0) + f(-2)$ if $f(x) = x^4 - x^2 + 1$.

5 Find $f(-3)$ if $f(x) = 2x^3 - 5x + 4$.

6 If $f(x) = 2x - 5$, find x when $f(x) = 13$.

7 Given $f(x) = x^2 + 3$, find any values of x for which $f(x) = 28$.

8 If $f(x) = 3^x$, find x when $f(x) = \frac{1}{27}$.

9 Find values of z for which $f(z) = 5$ given $f(z) = |2z + 3|$.

10 If $f(x) = 2x - 9$, find $f(p)$ and $f(x + h)$.

11 Find $g(x - 1)$ when $g(x) = x^2 + 2x + 3$.

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12 If $f(x) = x^2 - 1$, find $f(k)$ as a product of factors.

13 Given $f(t) = t^2 - 2t + 1$, find:

a t when $f(t) = 0$ **b** any values of t for which $f(t) = 9$.

14 Given $f(t) = t^4 + t^2 - 5$, find the value of $f(b) - f(-b)$.

15 $f(x) = \begin{cases} x^3 & \text{for } x > 1 \\ x & \text{for } x \le 1 \end{cases}$

Find $f(5)$, $f(1)$ and $f(-1)$.

16 $f(x) = \begin{cases} 2x - 4 & \text{if } x > 1 \\ x + 3 & \text{if } -1 \le x \le 1 \\ x^2 & \text{if } x < -1 \end{cases}$

Find the value of $f(2) - f(-2) + f(-1)$.

17 Find $g(3) + g(0) + g(-2)$ if $g(x) = \begin{cases} x + 1 & \text{when } x \ge 0 \\ -2x + 1 & \text{when } x < 0 \end{cases}$

18 Find the value of $f(3) - f(2) + 2f(-3)$ when $f(x) = \begin{cases} x & \text{for } x > 2 \\ x^2 & \text{for } -2 \le x \le 2 \\ 4 & \text{for } x < -2 \end{cases}$

19 Find the value of $f(-1) - f(3)$ if $f(x) = \begin{cases} x^3 - 1 & \text{for } x \ge 2 \\ 2x^2 + 3x - 1 & \text{for } x < 2 \end{cases}$

20 If $f(x) = x^2 - 5x + 4$, find $f(x + h) - f(x)$ in its simplest form.

21 Simplify $\dfrac{f(x+h) - f(h)}{h}$ where $f(x) = 2x^2 + x$.

22 If $f(x) = 5x - 4$, find $f(x) - f(c)$ in its simplest form.

23 Find the value of $f(k^2)$ if $f(x) = \begin{cases} 3x + 5 & \text{for } x \ge 0 \\ x^2 & \text{for } x < 0 \end{cases}$

24 If $f(x) = \begin{cases} x^3 & \text{when } x \ge 3 \\ 5 & \text{when } 0 < x < 3 \\ x^2 - x + 2 & \text{when } x \le 0 \end{cases}$

evaluate:

a $f(0)$ **b** $f(2) - f(1)$ **c** $f(-n^2)$

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25 If $f(x) = \dfrac{x^2 - 2x - 3}{x - 3}$:

a evaluate $f(2)$.

b explain why the function does not exist for $x = 3$.

c by taking several x values close to 3, find the value of y that the function is moving towards as x moves towards 3.

WS

Function notation

4.03 Properties of functions

We can use the properties of functions, such as their **intercepts**, to draw their graphs.

Intercepts

The ***x*-intercept** of a graph is the value of x where the graph crosses the x-axis.

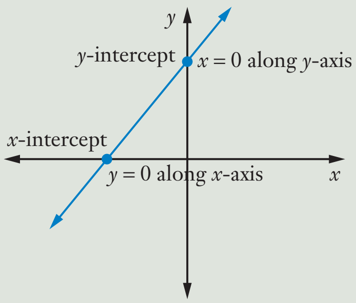
The ***y*-intercept** of a graph is the value of y where the graph crosses the y-axis.

Intercepts of the graph of a function

For x-intercept(s), substitute $y = 0$.

For y-intercept, substitute $x = 0$.

For the graph of $y = f(x)$, solving $f(x) = 0$ gives the x-intercepts and evaluating $f(0)$ gives the y-intercept.



EXAMPLE 7

Find the x- and y-intercepts of the function $f(x) = x^2 + 7x - 8$.

Solution

For x-intercepts, $y = f(x) = 0$:

$$0 = x^2 + 7x - 8$$
$$= (x + 8)(x - 1)$$
$$x = -8, x = 1$$

So x-intercepts are -8 and 1.

For y-intercept, $x = 0$:

$$f(0) = 0^2 + 7(0) - 8$$
$$= -8$$

So the y-intercept is -8.

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Domain and range

The **domain** of a function $y = f(x)$ is the set of all x values for which $f(x)$ is defined.

The **range** of a function $y = f(x)$ is the set of all y values for which $f(x)$ is defined.

Interval notation

- $[a, b]$ means the interval is between a and b, including a and b
- (a, b) means the interval is between a and b, excluding a and b
- $[a, b)$ means the interval is between a and b, including a but excluding b
- $(a, b]$ means the interval is between a and b, excluding a but including b
- $(-\infty, \infty)$ means that the interval includes the set of all real numbers R

EXAMPLE 8

Domain and range

Find the domain and range of each function.

a $f(x) = x^2$ **b** $y = \sqrt{x-1}$

Solution

a You can find the domain and range from the equation or the graph.

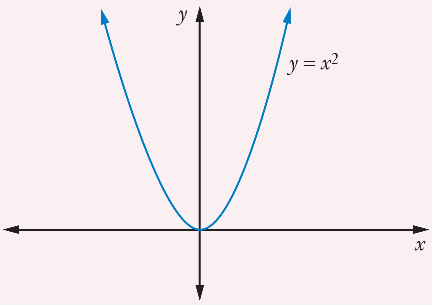
For $f(x) = x^2$, you can substitute any value for x. The y values will be 0 or positive.

So the domain is all real values of x and the range is all $y \geq 0$.

We can write this using interval notation:

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$



b The function $y = \sqrt{x-1}$ is only defined if $x - 1 \geq 0$ because we can only evaluate the square root of a positive number or 0.

For example, $x = 0$ gives $y = \sqrt{-1}$, which is undefined for real numbers.

So $x - 1 \geq 0$

$x \geq 1$

Domain: $[1, \infty)$

The value of $\sqrt{x-1}$ is always positive or zero. So $y \geq 0$.

Range: $[0, \infty)$

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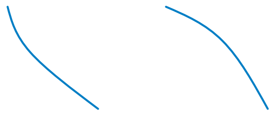
Increasing and decreasing graphs

When you draw a graph, it helps to know whether the function is increasing or decreasing on an interval.

If a graph is **increasing**, y increases as x increases, and the graph is moving upwards.

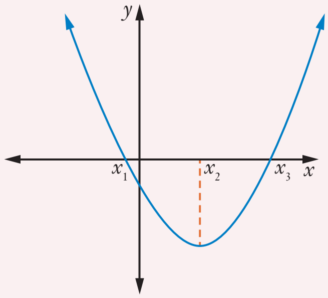
If a graph is **decreasing**, then y decreases as x increases, and the curve moves downwards.



EXAMPLE 9

State the domain over which each curve is increasing.

a



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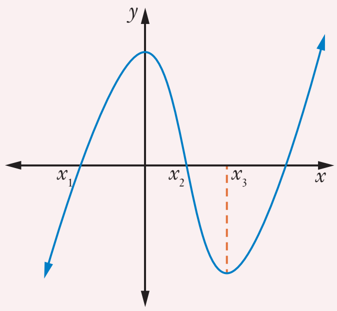

Solution

a The curve is decreasing to the left of x_2 and increasing to the right of x_2, that is, when $x > x_2$.

So the domain over which the graph is increasing is (x_2, ∞).

b The curve is increasing on the left of the y-axis ($x = 0$), decreasing from $x = 0$ to $x = x_3$, then increasing again from $x = x_3$.

So the curve is increasing for $x < 0$, $x > x_3$.

So the domain over which the graph is increasing is $(-\infty, 0) \cup (x_3, \infty)$.

The symbol $\cup$ is for 'union' and means 'and'. It stands for the union or joining of 2 separate parts. You will meet this symbol again in probability.

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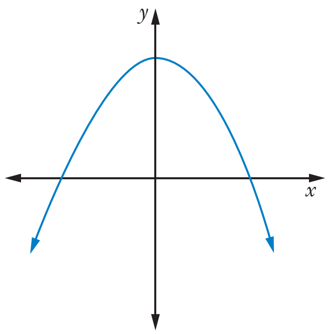
Even and odd functions

Even functions have graphs that are symmetrical about the y-axis. The graph has line symmetry about the y-axis. The left and right halves are mirror-images of each other.



Even functions

A function is even if $f(x) = f(-x)$ for all values of x in the domain.

EXAMPLE 10

Show that $f(x) = x^2 + 3$ is an even function.

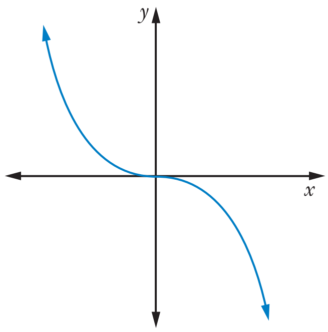
Odd and even functions

Solution

$$\begin{aligned} f(-x) &= (-x)^2 + 3 \\ &= x^2 + 3 \\ &= f(x) \end{aligned}$$

So $f(x) = x^2 + 3$ is an even function.

Odd functions have graphs that have point symmetry about the origin. A graph rotated 180° about the origin gives the original graph.



Odd functions

A function is odd if $f(-x) = -f(x)$ for all values of x in the domain.

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Odd and even functions

EXAMPLE 11

Show that $f(x) = x^3 - x$ is an odd function.

Solution

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$

So $f(x) = x^3 - x$ is an odd function.

INVESTIGATION

EVEN AND ODD FUNCTIONS

Explore the family of graphs of $f(x) = kx^n$, the **power functions**.

For what values of n is the function even?

For what values of n is the function odd?

Does the value of k change this?

Are these families of functions below even or odd? Does the value of k change this?

1 $f(x) = x^n + k$ **2** $f(x) = (x + k)^n$

Exercise 4.03 Properties of functions

1 Find the x- and y-intercepts of each function.

a $y = 3x - 2$ **b** $2x - 5y + 20 = 0$ **c** $x + 3y - 12 = 0$

d $f(x) = x^2 + 3x$ **e** $f(x) = x^2 - 4$ **f** $p(x) = x^2 + 5x + 6$

g $y = x^2 - 8x + 15$ **h** $p(x) = x^3 + 5$ **i** $y = \dfrac{x + 3}{x}$

j $g(x) = 9 - x^2$

2 $f(x) = 3x - 6$

a Solve $f(x) = 0$.

b Find the x- and y-intercepts.

3 Show that $f(x) = f(-x)$ where $f(x) = x^2 - 2$. What type of function is it?

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4 $f(x) = x^3 + 1$

a Find $f(x^2)$. **b** Find $[f(x)]^2$. **c** Find $f(-x)$.

d Is $f(x) = x^3 + 1$ an even or odd function?

e Solve $f(x) = 0$.

f Find the intercepts of the function.

5 Show that $g(x) = x^8 + 3x^4 - 2x^2$ is an even function.

6 Show that $f(x)$ is odd, given $f(x) = x$.

7 Show that $f(x) = x^2 - 1$ is an even function.

8 Show that $f(x) = 4x - x^3$ is an odd function.

9 **a** Prove that $f(x) = x^4 + x^2$ is an even function.

b Find $f(x) - f(-x)$.

10 Are these functions even, odd or neither?

a $y = \dfrac{x^3}{x^4 - x^2}$ **b** $f(x) = \dfrac{1}{x^3 - 1}$ **c** $f(x) = \dfrac{3}{x^2 - 4}$

d $y = \dfrac{x-3}{x+3}$ **e** $f(x) = \dfrac{x^3}{x^5 - x^2}$

11 If n is a positive integer, for what values of n is the power function $f(x) = kx^n$:

a even? **b** odd?

12 Can the function $f(x) = x^n + x$ ever be:

a even? **b** odd?

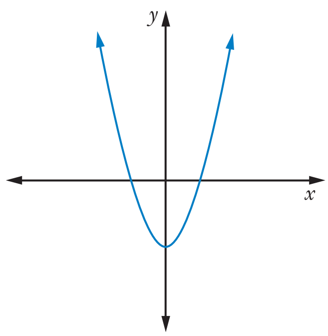
13 For the functions below, state:

i the domain over which the graph is increasing

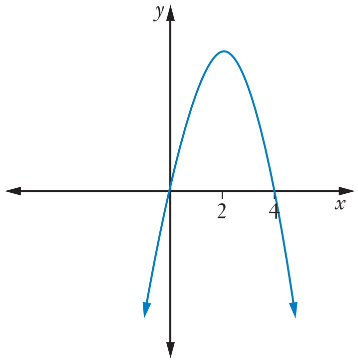
ii the domain over which the graph is decreasing

iii whether the graph is odd, even or neither.

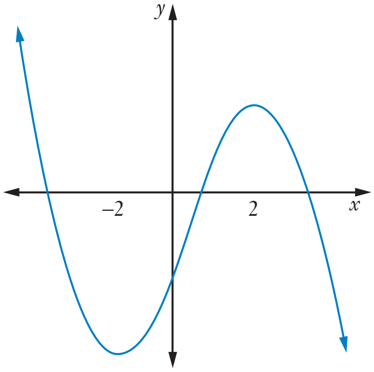
a



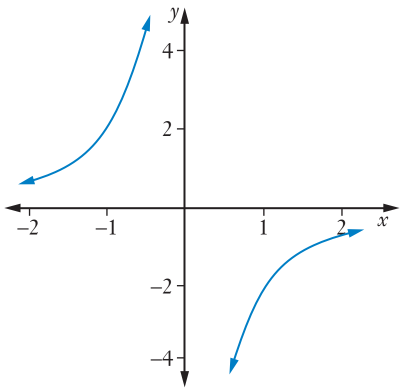
b



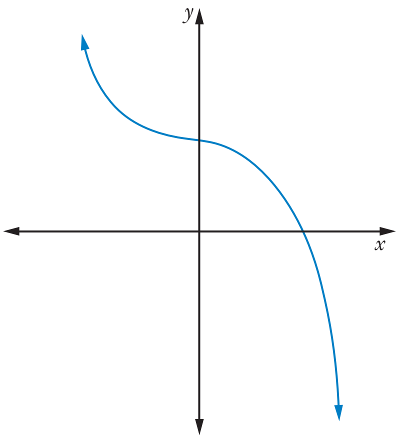
c



d



e



14 State the domain and range for each function.

a $f(x) = x^2 + 1$ **b** $y = x^3$ **c** $y = \sqrt{x}$

d $f(x) = \sqrt{x+5}$ **e** $y = -\sqrt{2x-6}$

15 $f(x) = (x - 2)^2$

a Find $f(3)$.

b Find $f(-5)$.

c Solve $f(x) = 0$.

d Find the x- and y-intercepts.

e State the domain and range of $f(x)$.

f Find $f(-x)$.

g Is $f(x)$ even, odd or neither?

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4.04 Linear functions

WS
A page of number planes

WS
Graphing linear functions

WS
x- and y-intercepts

Linear functions

A **linear function** has an equation of the form $y = mx + c$ or $ax + by + c = 0$.

Its graph is a straight line with one x-intercept and one y-intercept.

Direct variation

When one variable is in **direct variation** (or **direct proportion**) with another variable, one is a constant multiple of the other. This means that as one increases, so does the other.

Direct variation

If variables x and y are in direct proportion we can write the equation $y = kx$, where k is called the **proportionality constant**.

EXAMPLE 12

Huang earns \$20 an hour. Find an equation for Huang's income (I) for working x hours and draw its graph.

Solution

Income for 1 hour is \$20.

Income for 2 hours is \$20 × 2 or \$40.

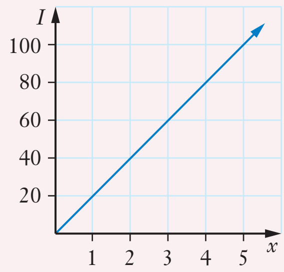
Income for 3 hours is \$20 × 3 or \$60.

Income for x hours is \$20 × x or \$20x.

We can write the equation as $I = 20x$.

We can graph the equation using a table of values.

x	1	2	3
I	20	40	60



$I = 20x$ is an example of direct variation. Direct variation graphs are always straight lines passing through the origin.

Graphing linear functions

EXAMPLE 13

a Find the x- and y-intercepts of the graph of $y = 2x - 4$ and draw its graph on the number plane.

b Find the x- and y-intercepts of the line with equation $x + 2y + 6 = 0$ and draw its graph.

Solution

a For x-intercept, $y = 0$:

$0 = 2x - 4$

$4 = 2x$

$2 = x$

So the x-intercept is 2.

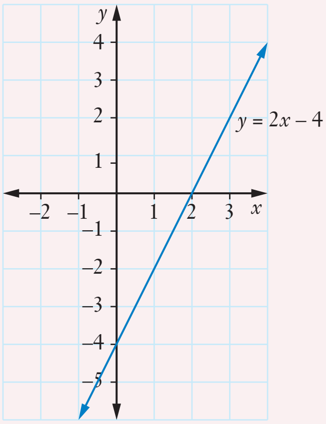
For y-intercept, $x = 0$:

$y = 2(0) - 4$

$= -4$

So the y-intercept is -4.

Use the intercepts to graph the line.



b For x-intercept, $y = 0$:

$x + 2(0) + 6 = 0$

$x + 6 = 0$

$x = -6$

So the x-intercept is -6.

For y-intercept, $x = 0$

$0 + 2y + 6 = 0$

$2y = -6$

$y = -3$

So the y-intercept is -3.

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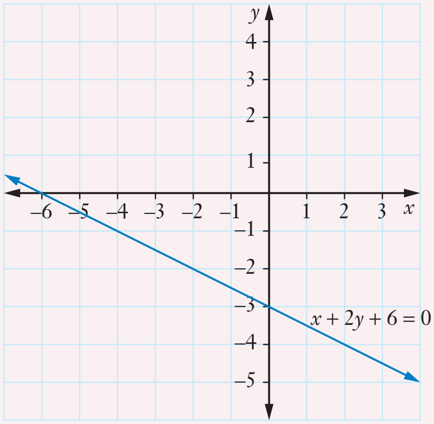


Domain and range of linear functions

The domain of a linear function is $(-\infty, \infty)$, all real numbers.

The range of a linear function is $(-\infty, \infty)$, all real numbers.

Horizontal and vertical lines

EXAMPLE 14

a Sketch the graph of $y = 2$ on a number plane. What is its domain and range?

b Sketch the graph of $x = -1$ on a number plane and state its domain and range.

Solution

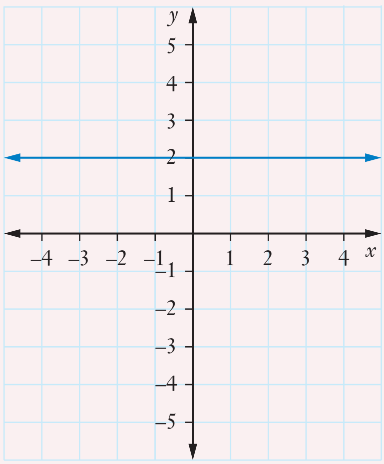
a x can have any value and y is always 2.

Some of the points on the line will be (0, 2), (1, 2) and (2, 2).

This gives a horizontal line with y-intercept 2.

The domain is all real x and the range is $y = 2$.

Domain $(-\infty, \infty)$, Range [2]



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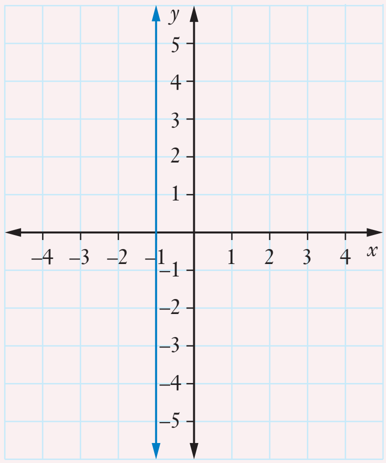
b y can have any value and x is always -1.

Some of the points on the line will be $(-1, 0)$, $(-1, 1)$ and $(-1, 2)$.

This gives a vertical line with x-intercept -1.

Domain $[-1]$, Range $(-\infty, \infty)$.



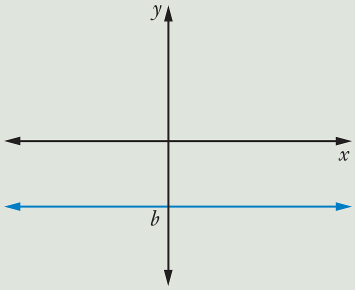
Horizontal lines

$y = b$ is a horizontal line with y-intercept b.

$y = b$ is a many-to-one function.

Domain $(-\infty, \infty)$

Range $[b]$



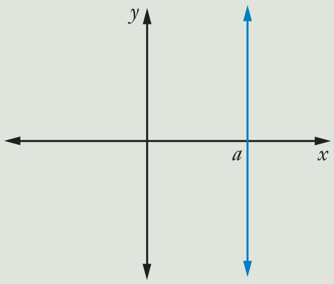
Vertical lines

$x = a$ is a vertical line with x-intercept a.

$x = a$ is not a function.

Domain $[a]$

Range $(-\infty, \infty)$



Exercise 4.04 Linear functions

1 Write an equation for:

a the number of months (N) in x years

b the amount of juice (A) in n lots of 2 litre bottles

c the cost (c) of x litres of petrol at \$1.50 per litre

d the number (y) of people in x debating teams if there are 4 people in each team

e the weight (w) of x lots of 400 g cans of peaches.

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2 Find the equation and draw the graph of the cost (c) of x refrigerators if each refrigerator costs \$850.

3 Find the x- and y-intercepts of the graph of each function.

a $y = x - 2$	**b** $y = 3x + 9$	**c** $y = 4 - 2x$
d $f(x) = 2x + 3$	**e** $f(x) = 5x - 4$	**f** $f(x) = 10x + 5$
g $x + y - 2 = 0$	**h** $2x - y + 4 = 0$	**i** $x - y + 3 = 0$
j $3x - 6y - 2 = 0$		

4 Draw the graph of each linear function.

a $y = x + 4$	**b** $f(x) = 2x - 1$	**c** $f(x) = 3x + 2$
d $x + y = 3$	**e** $x - y - 1 = 0$	

5 Find the domain and range of each equation.

a $3x - 2y + 7 = 0$	**b** $y = 2$	**c** $x = -4$
d $x - 2 = 0$	**e** $3 - y = 0$	

6 Sketch each equation's graph and state its domain and range.

a $x = 4$	**b** $x - 3 = 0$
c $y = 5$	**d** $y + 1 = 0$

7 A supermarket has boxes containing cans of dog food. The number of cans of dog food is directly proportional to the number of boxes.

- **a** If there are 144 cans in 4 boxes, find an equation for the number of cans (N) in x boxes.
- **b** How many cans are in 28 boxes?
- **c** How many boxes would be needed for 612 cans of dog food?

8 By sketching the graphs of $x - y - 4 = 0$ and $2x + 3y - 3 = 0$ on the same set of axes, find the point where they cross.

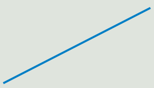
4.05 The gradient of a straight line

Gradient and y-intercept of a line

The **gradient** of a line measures its slope. It compares the vertical rise with the horizontal run.

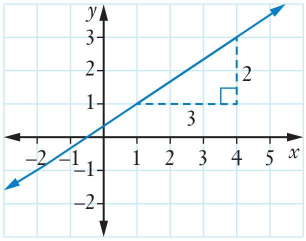
The gradient of a line

$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$



Positive gradient leans to the right.

Negative gradient leans to the left.



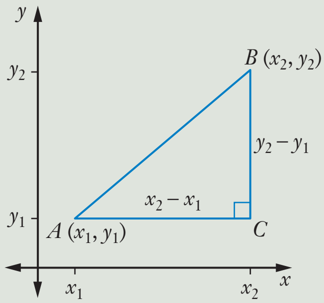
$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{2}{3}$$

On the number plane, gradient is a measure of the rate of change of y with respect to x.

Gradient formula

The gradient of the line joining points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



EXAMPLE 15

Find the gradient of the line joining points (2, 3) and (–3, 4).

Solution

$$\begin{aligned}
\text{Gradient } m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{4-3}{-3-2} \\
&= \frac{1}{-5} \\
&= -\frac{1}{5}
\end{aligned}$$

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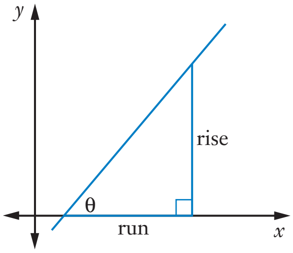
The angle of inclination of a line

The **angle of inclination**, θ, is the angle a straight line makes with the positive x-axis, measured anticlockwise.

$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{\text{opposite}}{\text{adjacent}}$$

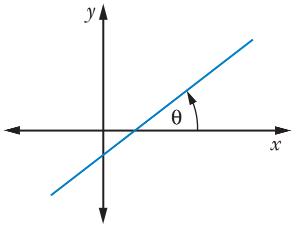
$$= \tan \theta$$



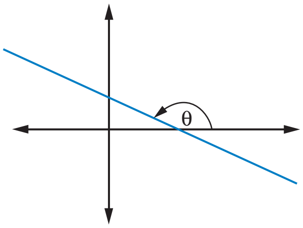
Gradient and angle of inclination of a line

$$m = \tan \theta$$

where m is the gradient and θ is the **angle of inclination**.



For an acute angle, $\tan \theta > 0$.



For an obtuse angle, $\tan \theta < 0$.

DISCUSSION

ANGLES AND GRADIENTS

1. What type of angles give a positive gradient?
2. What type of angles give a negative gradient? Why?
3. What is the gradient of a horizontal line? What angle does it make with the x-axis?
4. What angle does a vertical line make with the x-axis? Can you find its gradient?

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EXAMPLE 16

a Find the gradient of the line that makes an angle of inclination of 135°.

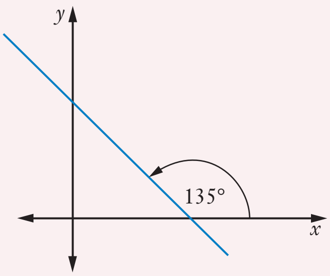
b Find correct to the nearest minute the angle of inclination of a straight line whose gradient is:

i 0.5 **ii** −3

Solution

a

$$\begin{aligned} m &= \tan\theta \\ &= \tan 135° \\ &= -1 \end{aligned}$$



b **i**

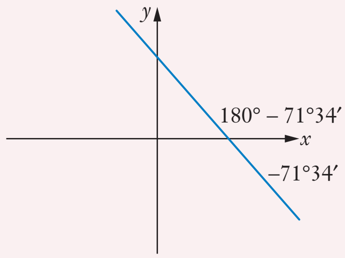
$$\begin{aligned} m &= \tan\theta \\ \therefore \tan\theta &= 0.5 \\ \theta &= \tan^{-1}(0.5) \\ &= 26°33'54.18'' \\ &\approx 26°34' \end{aligned}$$

See page 228 of Chapter 5 if you need to revise rounding an angle to the nearest minute.

Operation	Casio scientific	Sharp scientific
Enter data.	SHIFT tan⁻¹ 0.5 =	2ndF tan⁻¹ 0.5 =
Change to degrees and minutes.	° ' "	2ndF D°M'S

ii

$$\begin{aligned} m &= \tan\theta \\ \therefore \tan\theta &= -3 \\ \theta &= \tan^{-1}(-3) \\ &= -71°33'54.18'' \\ &\approx -71°34' \end{aligned}$$



A negative gradient means that the angle of inclination is obtuse.

To find this angle, subtract the acute angle from 180°.

$$\begin{aligned} \theta &= 180° - 71°34' \\ &= 108°26' \end{aligned}$$

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INVESTIGATION

GRAPHING $y = mx + c$

Graph each linear function using a graphics calculator or graphing software. Find the gradient of each function. What do you notice?

1 $y = x$	**2** $y = 2x$	**3** $y = 3x$	**4** $y = 4x$
5 $y = -x$	**6** $y = -2x$	**7** $y = -3x$	**8** $y = -4x$

Graph each function and find the y-intercept.

9 $y = x$	**10** $y = x + 1$	**11** $y = x + 2$	**12** $y = x + 3$
13 $y = x - 1$	**14** $y = x - 2$	**15** $y = x - 3$	

The gradient–intercept equation of a straight line

The linear function with equation $y = mx + c$ has gradient m and y-intercept c.

WS

$y = mx + c$

EXAMPLE 17

a Find the gradient and y-intercept of the linear function $y = 7x - 5$.

b Find the gradient of the straight line with equation $2x + 3y - 6 = 0$.

Solution

a Gradient = 7, y-intercept = −5.

b First, change the equation into the form $y = mx + c$.

$$\begin{aligned} 2x + 3y - 6 &= 0 \\ 2x + 3y &= 6 \\ 3y &= 6 - 2x \\ &= -2x + 6 \end{aligned}$$

$$\begin{aligned} y &= \frac{-2x}{3} + \frac{6}{3} \\ &= -\frac{2}{3}x + 2 \end{aligned}$$

So the gradient is $-\frac{2}{3}$.

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Exercise 4.05 The gradient of a straight line

1 Find the gradient of the line joining the points:

a $(3, 2)$ and $(1, -2)$ **b** $(0, 2)$ and $(3, 6)$ **c** $(-2, 3)$ and $(4, -5)$

d $(2, -5)$ and $(-3, 7)$ **e** $(2, 3)$ and $(-1, 1)$ **f** $(-5, 1)$ and $(3, 0)$

g $(-2, -3)$ and $(-4, 6)$ **h** $(-1, 3)$ and $(-7, 7)$ **i** $(1, -4)$ and $(5, 5)$

2 Find the gradient of the straight line, correct to 1 decimal place, whose angle of inclination is:

a $25°$ **b** $82°$ **c** $68°$

d $100°$ **e** $130°$ **f** $164°$

3 For each linear function, find:

i the gradient **ii** the y-intercept.

a $y = 3x + 5$ **b** $f(x) = 2x + 1$ **c** $y = 6x - 7$

d $y = -x$ **e** $y = -4x + 3$ **f** $y = x - 2$

g $f(x) = 6 - 2x$ **h** $y = 1 - x$ **i** $y = 9x$

4 Find the gradient of the linear function:

a with x-intercept 3 and y-intercept -1

b passing through $(2, 4)$ and x-intercept 5

c passing through $(1, 1)$ and $(-2, 7)$

d with x-intercept -3 and passing through $(2, 3)$

e passing through the origin and $(-3, -1)$.

5 Find the angle of inclination, to the nearest minute, of a line with gradient:

a 2 **b** 1.7 **c** 6

d -5 **e** -0.85 **f** -1.2

6 For each linear function, find:

i the gradient **ii** the y-intercept.

a $2x + y - 3 = 0$ **b** $5x + y + 6 = 0$ **c** $6x - y - 1 = 0$

d $x - y + 4 = 0$ **e** $4x + 2y - 1 = 0$ **f** $6x - 2y + 3 = 0$

g $x + 3y + 6 = 0$ **h** $4x + 5y - 10 = 0$ **i** $7x - 2y - 1 = 0$

7 Find the gradient of each linear function.

a $y = -2x - 1$ **b** $y = 2$ **c** $x + y + 1 = 0$

d $3x + y = 8$ **e** $2x - y + 5 = 0$ **f** $x + 4y - 12 = 0$

g $3x - 2y + 4 = 0$ **h** $5x - 4y = 15$ **i** $y = \frac{2}{3}x + 3$

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j $y = \frac{x}{5} - 1$ **k** $y = \frac{2x}{7} + 5$ **l** $y = -\frac{3x}{5} - 2$

m $2y = -\frac{x}{7} + \frac{1}{3}$ **n** $3x - \frac{y}{5} = 8$ **o** $\frac{x}{2} + \frac{y}{3} = 1$

8 If the gradient of the line joining $(8, y_1)$ and $(-1, 3)$ is 2, find the value of y_1.

9 The gradient of the line through $(2, -1)$ and $(x, 0)$ is -5. Find the value of x.

10 The gradient of a line is -1 and the line passes through the points $(4, 2)$ and $(x, -3)$. Find the value of x.

11 The number of frequent flyer points that Mario earns on his credit card is directly proportional to the amount of money he spends on his card.

- **a** If Mario earns 150 points when he spends \$450, find an equation for the number of points (P) he earns when spending d dollars.
- **b** Find the number of points Mario earns when he spends \$840.
- **c** If Mario earns 57 points, how much did he spend?

12 The points $A(-1, 2)$, $B(1, 5)$, $C(6, 5)$ and $D(4, 2)$ form a parallelogram. Find the gradients of all 4 sides of the parallelogram. What do you notice?

4.06 Finding a linear equation

Linear functions code puzzle

Linear modelling

Finding the equation of a line

Equations of lines

EXAMPLE 18

Find the equation of the line with gradient 3 and y-intercept -1.

Solution

The equation is $y = mx + c$ where m = gradient and c = y-intercept.

$m = 3$ and $c = -1$.

Equation is $y = 3x - 1$.

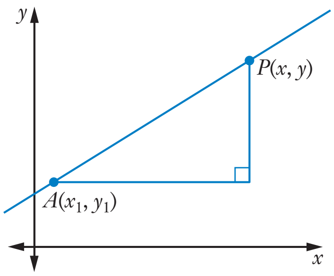
There is a formula you can use if you know the gradient and the coordinates of a point on the line.

The point–gradient equation of a straight line

The linear function with equation $y - y_1 = m(x - x_1)$ has gradient m and the point (x_1, y_1) lies on the line.

Proof

Let $P(x, y)$ be a general point on the line with gradient m that passes through $A(x_1, y_1)$.



Then line AP has gradient

$$m = \frac{y - y_1}{x - x_1}$$

$$m(x - x_1) = y - y_1$$

EXAMPLE 19

Find the equation of the line:

a with gradient -4 and x-intercept 1

b passing through $(2, 3)$ and $(-1, 4)$.

Solution

a The x-intercept of 1 means the line passes through the point $(1, 0)$.

Substituting $m = -4$, $x_1 = 1$ and $y_1 = 0$ into the formula:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -4(x - 1) \\ y &= -4x + 4 \end{aligned}$$

b First find the gradient.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 3}{-1 - 2} \\ &= -\frac{1}{3} \end{aligned}$$

Substitute the gradient and one of the points, say $(2, 3)$, into the formula.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -\frac{1}{3}(x - 2) \\ 3 \times (y - 3) &= 3 \times -\frac{1}{3}(x - 2) \\ 3y - 9 &= -(x - 2) \\ &= -x + 2 \\ x + 3y - 9 &= 2 \\ x + 3y - 11 &= 0 \end{aligned}$$

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Applications of linear functions

EXAMPLE 20

A solar panel company has fixed overhead costs of \$3000 per day and earns \$150 for each solar cell sold.

a Write the amount (\$$A$) that the company earns on selling x solar cells each day.

b Find the amount the company earns on a day when it sells 54 solar cells.

c If the company earns \$2850 on another day, how many solar cells did it sell that day?

d What is the **break-even point** for this company (where income and costs of production are equal)?

Solution

a The company earns \$150 per cell, so it earns \$$150x$ for x cells.

Daily amount earned = value of solar cells sold – overhead costs.

So $A = 150x - 3000$

b Substitute $x = 54$:

$A = 150(54) - 3000 = 5100$

The company earns \$5100 when it sells 54 solar cells.

c Substitute $A = 2850$:

$2850 = 150x - 3000$

$5850 = 150x$

$39 = x$

The company sold 39 solar cells that day.

d At the break-even point:

Income = overhead costs

$150x = 3000$ (or $A = 0$)

$x = 20$

So the break-even point is where the company sells 20 solar cells.

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Exercise 4.06 Finding a linear equation

1 Find the equation of the straight line:

- **a** with gradient 4 and y-intercept -1
- **b** with gradient -3 and passing through $(0, 4)$
- **c** passing through the origin with gradient 5
- **d** with gradient 4 and x-intercept -5
- **e** with x-intercept 1 and y-intercept 3
- **f** with x-intercept 3, y-intercept -4.

2 Find the equation of the straight line passing through the points:

- **a** $(2, 5)$ and $(-1, 1)$ **b** $(0, 1)$ and $(-4, -2)$ **c** $(-2, 1)$ and $(3, 5)$
- **d** $(3, 4)$ and $(-1, 7)$ **e** $(-4, -1)$ and $(-2, 0)$.

3 What is **a** the gradient and **b** the equation of the line with x-intercept 2 that passes through $(3, -4)$?

4 Find the equation of the line:

- **a** parallel to the x-axis and passing through $(2, 3)$
- **b** parallel to the y-axis and passing through $(-1, 2)$.

5 A straight line passing through the origin has a gradient of -2. Find:

- **a** the y-intercept **b** its equation.

6 In a game, each person starts with 20 points, then earns 15 points for every level completed.

- **a** Write an equation for the number of points earned (P) for x levels completed.
- **b** Find the number of points earned for completing:
  - **i** 24 levels **ii** 55 levels **iii** 247 levels
- **c** Find the number of levels completed if the number of points earned is:
  - **i** 2195 **ii** 7700 **iii** 12 665

7 A TV manufacturing business has fixed costs of \$1500 rental, \$3000 wages and other costs of \$2500 each week. It costs \$250 to produce each TV.

- **a** Write an equation for the cost (c) of producing n TVs each week.
- **b** From the equation, find the cost of producing:
  - **i** 100 TVs **ii** 270 TVs **iii** 1200 TVs
- **c** From the equation find the number of TVs produced if the cost is:
  - **i** \$52 000 **ii** \$78 250 **iii** \$367 000
- **d** If each TV sells for \$950, find the number of TVs needed to sell to break even.

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8 There are 450 litres of water in a pond, and 8 litres of water evaporate out of the pond every hour.

a Write an equation for the amount of water in the pond (A) after h hours.

b Find the amount of water in the pond after:

i 3 hours **ii** a day.

c After how many hours will the pond be empty?

9 Geordie has a \$20 iTunes credit. He uses the credit to buy singles at \$1.69 each.

a Write an equation for the amount of credit (C) left if Geordie buys x singles.

b How many songs can Geordie buy before his credit runs out?

10 Emily-Rose owes \$20 000 and she pays back \$320 a month.

a Write an equation for the amount of money she owes (A) after x months.

b How much does Emily-Rose owe after:

i 5 months? **ii** 1 year? **iii** 5 years?

c How long will it take for Emily-Rose to pay all the money back?

11 Acme Party Supplies earns \$5 for every helium balloon it sells.

a If overhead costs are \$100 each day, find an equation for the profit (P) of selling x balloons.

b How much profit does Acme make if it sells 300 balloons?

c How many balloons does it sell if it makes a profit of \$1055?

d What is the break-even point for this business?

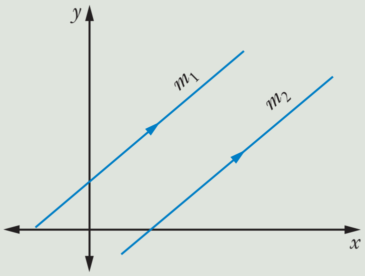
4.07 Parallel and perpendicular lines

WS Parallel and perpendicular lines

WS Linear functions

Gradients of parallel lines

If 2 lines are parallel, then they have the same gradient. That is, $m_1 = m_2$.



EXAMPLE 21

a Prove that the straight lines with equations $5x - 2y - 1 = 0$ and $5x - 2y + 7 = 0$ are parallel.

b Find the equation of a straight line parallel to the line $2x - y - 3 = 0$ and passing through $(1, -5)$.

Solution

a First, change the equation into the form $y = mx + c$.

$$5x - 2y - 1 = 0$$
$$5x - 1 = 2y$$
$$\frac{5}{2}x - \frac{1}{2} = y$$
$$\therefore \quad m_1 = \frac{5}{2}$$

$$5x - 2y + 7 = 0$$
$$5x + 7 = 2y$$
$$\frac{5}{2}x + \frac{7}{2} = y$$
$$\therefore \quad m_2 = \frac{5}{2}$$

$$m_1 = m_2 = \frac{5}{2}$$

$\therefore$ the lines are parallel.

b $$2x - y - 3 = 0$$
$$2x - 3 = y$$
$$\therefore m_1 = 2$$

For parallel lines $m_1 = m_2$.

$$\therefore m_2 = 2$$

Substitute this and $(1, -5)$ into $y - y_1 = m(x - x_1)$:

$$y - (-5) = 2(x - 1)$$
$$y + 5 = 2x - 2$$
$$y = 2x - 7$$

CLASS INVESTIGATION

PERPENDICULAR LINES

Sketch each pair of straight lines on the same number plane.

1 $3x - 4y + 12 = 0$ and $4x + 3y - 8 = 0$

2 $2x + y + 4 = 0$ and $x - 2y + 2 = 0$

What do you notice about each pair of lines?

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Gradients of perpendicular lines

If 2 lines with gradients m_1 and m_2 are perpendicular, then $m_1m_2 = -1$,

that is, $m_2 = -\frac{1}{m_1}$.

EXAMPLE 22

a Show that the lines with equations $3x + y - 11 = 0$ and $x - 3y + 1 = 0$ are perpendicular.

b Find the equation of the straight line through (2, 3) that is perpendicular to the line passing through (−1, 7) and (3, 3).

Solution

a

$$3x + y - 11 = 0$$
$$y = -3x + 11$$
$$\therefore m_1 = -3$$
$$x - 3y + 1 = 0$$
$$x + 1 = 3y$$
$$\frac{1}{3}x + \frac{1}{3} = y$$
$$\therefore m_2 = \frac{1}{3}$$
$$m_1m_2 = -3 \times \frac{1}{3} = -1$$

∴ the lines are perpendicular.

b Line through (−1, 7) and (3, 3):

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{7-3}{-1-3}$$
$$= \frac{4}{-4}$$
$$= -1$$

For perpendicular lines, $m_1m_2 = -1$:

$$-1m_2 = -1$$
$$m_2 = 1$$

Substitute $m = 1$ and the point (2, 3) into $y - y_1 = m(x - x_1)$:

$$y - 3 = 1(x - 2)$$
$$= x - 2$$
$$y = x + 1$$

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Exercise 4.07 Parallel and perpendicular lines

1 Find the gradient of the straight line:

- **a** parallel to the line $3x + y - 4 = 0$
- **b** perpendicular to the line $3x + y - 4 = 0$
- **c** parallel to the line joining $(3, 5)$ and $(-1, 2)$
- **d** perpendicular to the line with x-intercept 3 and y-intercept 2
- **e** perpendicular to the line that has an angle of inclination of $135°$
- **f** perpendicular to the line $6x - 5y - 4 = 0$
- **g** parallel to the line $x - 3y - 7 = 0$
- **h** perpendicular to the line passing through $(4, -2)$ and $(3, 3)$.

2 Find the equation of the straight line:

- **a** passing through $(2, 3)$ and parallel to the line $y = x + 6$
- **b** through $(-1, 5)$ and parallel to the line $x - 3y - 7 = 0$
- **c** with x-intercept 5 and parallel to the line $y = 4 - x$
- **d** through $(3, -4)$ and perpendicular to the line $y = 2x$
- **e** through $(-2, 1)$ and perpendicular to the line $2x + y + 3 = 0$
- **f** through $(7, -2)$ and perpendicular to the line $3x - y - 5 = 0$
- **g** through $(-3, -1)$ and perpendicular to the line $4x - 3y + 2 = 0$
- **h** passing through the origin and parallel to the line $x + y + 3 = 0$
- **i** through $(3, 7)$ and parallel to the line $5x - y - 2 = 0$
- **j** through $(0, -2)$ and perpendicular to the line $x - 2y = 9$
- **k** perpendicular to the line $3x + 2y - 1 = 0$ and passing through the point $(-2, 4)$.

3 Show that the lines with equations $y = 3x - 2$ and $6x - 2y - 9 = 0$ are parallel.

4 Show that lines $x + 5y = 0$ and $y = 5x + 3$ are perpendicular.

5 Show that lines $6x - 5y + 1 = 0$ and $6x - 5y - 3 = 0$ are parallel.

6 Show that lines $7x + 3y + 2 = 0$ and $3x - 7y = 0$ are perpendicular.

7 If the lines $3x - 2y + 5 = 0$ and $y = kx - 1$ are perpendicular, find the value of k.

8 Show that the line joining $(3, -1)$ and $(2, -5)$ is parallel to the line $8x - 2y - 3 = 0$.

9 Show that the points A$(-3, -2)$, $B(-1, 4)$, $C(7, -1)$ and $D(5, -7)$ are the vertices of a parallelogram.

10 The points $A(-2, 0)$, $B(1, 4)$, $C(6, 4)$ and $D(3, 0)$ form a rhombus. Show that the diagonals are perpendicular.

11 Find the equation of the straight line passing through $(6, -3)$ that is perpendicular to the line joining $(2, -1)$ and $(-5, -7)$.

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4.08 Quadratic functions

WS Graphing quadratic functions

WS Graphing quadratics

Quadratic functions

A **quadratic function** has an equation in the form $y = ax^2 + bx + c$, where the highest power of x is 2. The graph of a quadratic function is a **parabola**.

EXAMPLE 23

Graph the quadratic function $y = x^2 - x$.

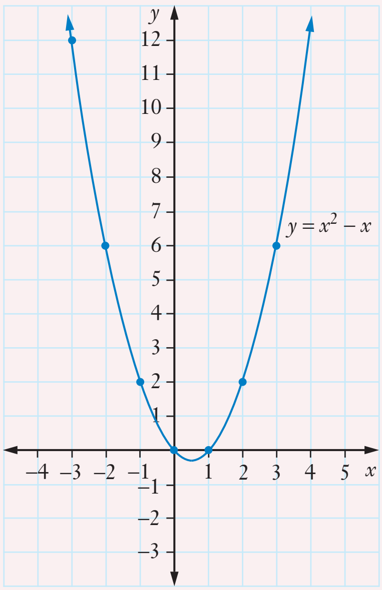
Solution

Draw up a table of values for $y = x^2 - x$.

x	–3	–2	–1	0	1	2	3
y	12	6	2	0	0	2	6

Plot (–3, 12), (–2, 6), (–1, 2), (0, 0), (1, 0), (2, 2) and (3, 6) and draw a parabola through them.

Label the graph with its equation.



TECHNOLOGY

Transforming quadratic functions

Use a graphics calculator or graphing software to graph these quadratic functions. Look for any patterns.

$y = x^2$	$y = x^2 + 1$	$y = x^2 + 2$	$y = x^2 + 3$
$y = x^2 - 1$	$y = x^2 - 2$	$y = x^2 - 3$	$y = 2x^2$
$y = 3x^2$	$y = x^2 + x$	$y = x^2 + 2x$	$y = x^2 + 3x$
$y = x^2 - x$	$y = x^2 - 2x$	$y = x^2 - 3x$	$y = -x^2$
$y = -x^2 + 1$	$y = -x^2 + 2$	$y = -x^2 + 3$	$y = -x^2 - 1$
$y = -x^2 - 2$	$y = -x^2 - 3$	$y = -2x^2$	$y = -3x^2$
$y = -x^2 + x$	$y = -x^2 + 2x$	$y = -x^2 - x$	$y = -x^2 - 2x$

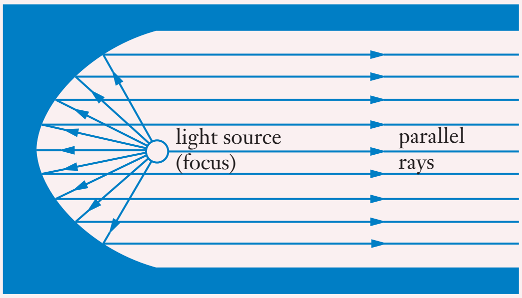
Could you predict where the graphs $y = x^2 + 9$, $y = 5x^2$ or $y = x^2 + 6x$ would lie?

Is the parabola always a function? Can you find an example of a parabola that is not a function?

DID YOU KNOW?

The parabola

The parabola shape has special properties that are very useful. For example, if a light is placed inside a parabolic mirror at a special place called the focus, then all light rays coming from this point and reflecting off the parabola shape will radiate out parallel to each other, giving a strong light. This is how car headlights work. The dishes of radio telescopes also use this property of the parabola, because radio signals coming in to the dish will reflect back to the focus.



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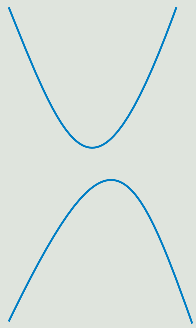
Concavity and turning points

For the parabola $y = ax^2 + bx + c$:

- if $a > 0$ the parabola is **concave upwards** and has a **minimum turning point**.
- if $a < 0$ the parabola is **concave downwards** and has a **maximum turning point**.



The **turning point** is also called the **vertex** or **stationary point** of the parabola.

Notice also that the parabola is always symmetrical.

EXAMPLE 24

a **i** Sketch the graph of $y = x^2 - 1$, showing intercepts.

ii State the domain and range.

b **i** Find the x- and y-intercepts of the quadratic function $f(x) = -x^2 + 4x + 5$.

ii Sketch a graph of the function.

iii Find the maximum value of the function.

iv State the domain and range.

Solution

a **i** Since $a > 0$, the graph is concave upwards.

For x-intercepts, $y = 0$:

$0 = x^2 - 1$

$1 = x^2$

$x = \pm 1$

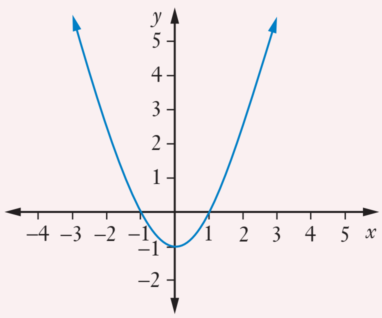
For y-intercept, $x = 0$:

$y = 0^2 - 1$

$= -1$

Since the parabola is symmetrical, the turning point is at $x = 0$, halfway between the x-intercepts -1 and 1.

When $x = 0$, $y = -1$: Vertex is $(0, -1)$.



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ii From the equation and the graph, x can have any value.

Domain $(-\infty, \infty)$

The values of y are greater than or equal to -1.

Range $[-1, \infty)$

b **i** For x-intercepts, $f(x) = 0$.

$0 = -x^2 + 4x + 5$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

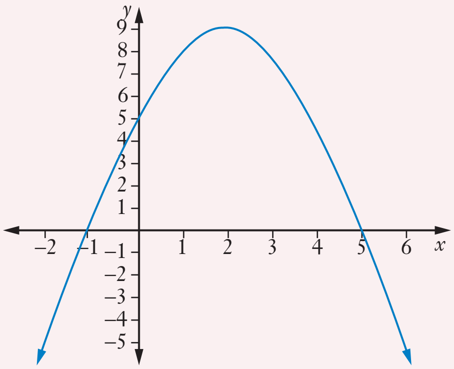
$x = 5, x = -1$

For y-intercept, $x = 0$.

$f(0) = -(0)^2 + 4(0) + 5$

$= 5$

ii Since $a < 0$, the quadratic function is concave downwards.



iii The turning point is halfway between $x = -1$ and $x = 5$.

$$x = \frac{-1+5}{2}$$

$$= 2$$

$$f(2) = -(2)^2 + 4(2) + 5$$

$$= 9$$

The maximum value of $f(x)$ is 9.

iv For the domain, the function can take on all real numbers for x.

Domain $(-\infty, \infty)$

For the range, $y \le 9$.

Range $(-\infty, 9]$

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Exercise 4.08 Quadratic functions

1 Find the x- and y-intercepts of the graph of each quadratic function:

a $y = x^2 + 2x$ **b** $y = -x^2 + 3x$ **c** $f(x) = x^2 - 1$

d $y = x^2 - x - 2$ **e** $y = x^2 - 9x + 8$

2 Sketch each parabola and find its maximum or minimum value:

a $y = x^2 + 2$ **b** $y = -x^2 + 1$ **c** $f(x) = x^2 - 4$

d $y = x^2 + 2x$ **e** $y = -x^2 - x$ **f** $f(x) = (x - 3)^2$

g $f(x) = (x + 1)^2$ **h** $y = x^2 + 3x - 4$ **i** $y = 2x^2 - 5x + 3$

j $f(x) = -x^2 + 3x - 2$

3 For each parabola, find:

i the x- and y-intercepts **ii** the domain and range.

a $y = x^2 - 7x + 12$ **b** $f(x) = x^2 + 4x$ **c** $y = x^2 - 2x - 8$

d $y = x^2 - 6x + 9$ **e** $f(x) = 4 - x^2$

4 Find the domain and range of:

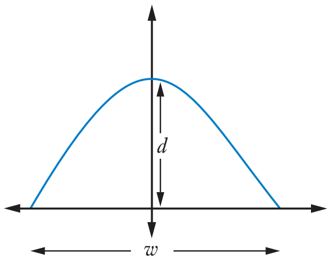
a $y = x^2 - 5$ **b** $f(x) = x^2 - 6x$ **c** $f(x) = x^2 - x - 2$

d $y = -x^2$ **e** $f(x) = (x - 7)^2$

5 A satellite dish is in the shape of a parabola with equation $y = -3x^2 + 6$, and all dimensions are in metres.

a Find d, the depth of the dish.

b Find w, the width of the dish, to 1 decimal place.



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WS Quadratic functions

WS Sketching quadratic functions

WS Features of a parabola

4.09 Axis of symmetry

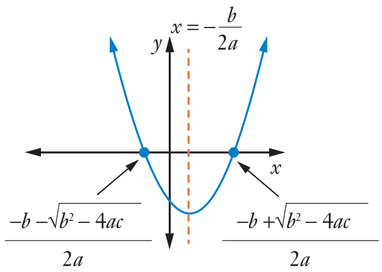
Axis of symmetry of a parabola

The **axis of symmetry** of a parabola with the equation $y = ax^2 + bx + c$ is the vertical line with equation:

$$x = -\frac{b}{2a}.$$

Proof

The axis of symmetry of a parabola lies halfway between the x-intercepts.



For the x-intercepts, $y = 0$

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The x-coordinate of the axis of symmetry is the average of the x-intercepts.

$$x = \frac{\dfrac{-b - \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}}{2} = \frac{\dfrac{-2b}{2a}}{2} = \frac{-2b}{4a} = -\frac{b}{2a}$$

Turning point of a parabola

The quadratic function $y = ax^2 + bx + c$ has a minimum value if $a > 0$ and a maximum value if $a < 0$.

The minimum or maximum value of the quadratic function is $f\left(-\frac{b}{2a}\right)$.

The turning point or vertex of a parabola is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

EXAMPLE 25

a Find the equation of the axis of symmetry and the minimum value of the quadratic function $y = x^2 - 5x + 1$.

b Find the equation of the axis of symmetry, the maximum value and the turning point of the quadratic function $y = -3x^2 + x - 5$.

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Solution

a Axis of symmetry:

$$x = -\frac{b}{2a}$$
$$= -\frac{(-5)}{2(1)}$$
$$= \frac{5}{2}$$
$$= 2\frac{1}{2}$$

$\therefore$ Axis of symmetry is the line $x = 2\frac{1}{2}$.

Minimum value $y = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1$

$$= \frac{25}{4} - \frac{25}{2} + 1$$
$$= -5\frac{1}{4}$$

b $x = -\frac{b}{2a}$

$$= -\frac{1}{2(-3)}$$
$$= \frac{1}{6}$$

$\therefore$ Axis of symmetry is the line $x = \frac{1}{6}$.

Maximum value $y = -3\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right) - 5$

$$= -\frac{1}{12} + \frac{1}{6} - 5$$
$$= -4\frac{11}{12}$$

The turning point is $\left(\frac{1}{6}, -4\frac{11}{12}\right)$.

EXAMPLE 26

Determine whether each function is even.

a $f(x) = x^2 + 3$

b $y = -x^2 + 3x$

Solution

a $f(x) = x^2 + 3$

$$f(-x) = (-x)^2 + 3$$
$$= x^2 + 3$$
$$= f(x)$$

So $f(x) = x^2 + 3$ is an even function.

b Let $f(x) = -x^2 + 3x$

$$f(-x) = -(-x)^2 + 3(-x)$$
$$= -x^2 - 3x$$
$$\neq f(x)$$

So $y = -x^2 + 3x$ is not an even function.

Exercise 4.09 Axis of symmetry

1 For the parabola $y = x^2 + 2x$, find the equation of its axis of symmetry and the minimum value.

2 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 - 4$.

3 Find the equation of the axis of symmetry and the minimum turning point of the parabola $y = 4x^2 - 3x + 1$.

4 Find the equation of the axis of symmetry and the maximum value of the parabola $y = -x^2 + 2x - 7$.

5 Find the equation of the axis of symmetry and the vertex of the parabola $y = -2x^2 - 4x + 5$.

6 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 + 3x + 2$.

7 Find the equation of the axis of symmetry and the coordinates of the vertex for each parabola:

a $y = x^2 + 6x - 3$ **b** $y = -x^2 - 8x + 1$ **c** $y = 3x^2 + 18x + 4$
d $y = -2x^2 + 5x$ **e** $y = 4x^2 + 10x - 7$

8 For each parabola, find:

i the equation of the axis of symmetry
ii the minimum or maximum value
iii the vertex.

a $y = x^2 + 2x - 2$ **b** $y = -2x^2 + 4x - 1$

9 Find the turning point of each function and state whether it is a maximum or minimum.

a $y = x^2 + 2x + 1$ **b** $y = x^2 - 8x - 7$ **c** $f(x) = x^2 + 4x - 3$
d $y = x^2 - 2x$ **e** $f(x) = x^2 - 4x - 7$ **f** $f(x) = 2x^2 + x - 3$
g $y = -x^2 - 2x + 5$ **h** $y = -2x^2 + 8x + 3$ **i** $f(x) = -3x^2 + 3x + 7$

10 For each quadratic function:

i find x-intercepts using the quadratic formula
ii state whether the function has a maximum or minimum value and find this value
iii sketch the graph of the function on a number plane
iv solve the quadratic equation $f(x) = 0$ graphically

a $f(x) = x^2 + 4x + 4$ **b** $f(x) = x^2 - 2x - 3$ **c** $y = x^2 - 6x + 1$
d $f(x) = -x^2 - 2x + 6$ **e** $f(x) = -x^2 - x + 3$

11 a Find the minimum value of the parabola with equation $y = x^2 - 2x + 5$.
b How many solutions does the quadratic equation $x^2 - 2x + 5 = 0$ have?
c Sketch the parabola.

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12 **a** Find the maximum value of the quadratic function $f(x) = -2x^2 + x - 4$.

b How many solutions are there to the quadratic equation $-2x^2 + x - 4 = 0$?

c Sketch the graph of the quadratic function.

13 Show that $f(x) = -x^2$ is an even function.

14 Determine which of these functions are even.

a $y = x^2 + 1$ **b** $f(x) = x^2 - 3$ **c** $y = -2x^2$

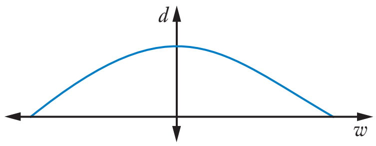
d $f(x) = x^2 - 3x$ **e** $f(x) = x^2 + x$ **f** $y = x^2 - 4$

g $y = x^2 - 2x - 3$ **h** $y = x^2 - 5x + 4$ **i** $p(x) = (x + 1)^2$

15 A bridge has a parabolic span as shown,

with equation $d = -\dfrac{w^2}{800} + 200$

where d is the depth of the arch in metres.



a Show that the quadratic function is even.

b Find the depth of the arch from the top of the span.

c Find the total width of the span.

d Find the depth of the arch at a point 10 m from its widest span.

e Find the width across the span at a depth of 100 m.

EXT1 4.10 Quadratic inequalities

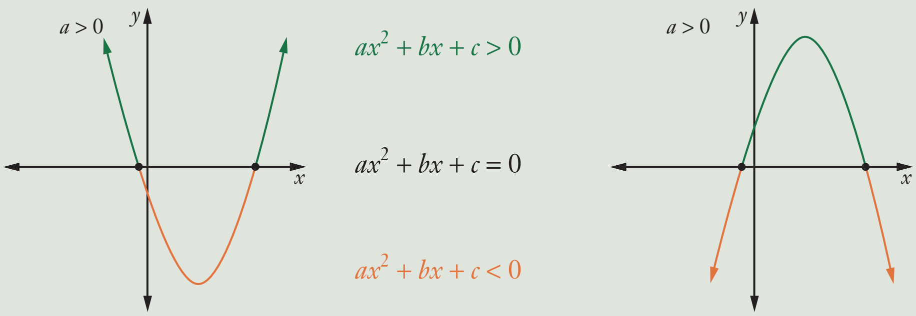
WS Quadratic inequalities

In Chapter 2, *Equations and inequalities*, you solved quadratic inequalities using the number line. You can also solve quadratic inequalities using the graph of a parabola.

The parabola and quadratic inequalities

For the graph of the quadratic function $y = ax^2 + bx + c$:

- $ax^2 + bx + c = 0$ on the x-axis
- $ax^2 + bx + c > 0$ above the x-axis
- $ax^2 + bx + c < 0$ below the x-axis



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Quadratic inequalities

EXAMPLE 27

Solve:

a $x^2 - 3x + 2 \geq 0$

b $4x - x^2 > 0$

Solution

a Sketch the graph of $y = x^2 - 3x + 2$ showing x-intercepts.

$a > 0$ so it is concave upwards.

For x-intercepts, $y = 0$.

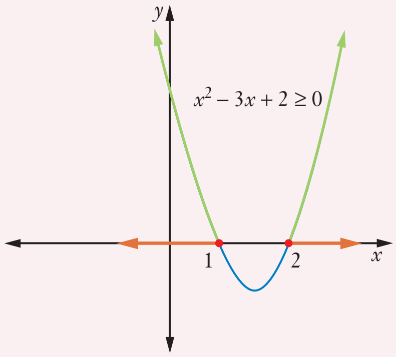
$0 = x^2 - 3x + 2$

$= (x - 2)(x - 1)$

$x = 2, x = 1$

$x^2 - 3x + 2 \geq 0$ on and above the x-axis.

$\therefore\ x \leq 1, x \geq 2$



b For $y = 4x - x^2$, $a < 0$ so its graph is concave downwards.

For x-intercepts, $y = 0$:

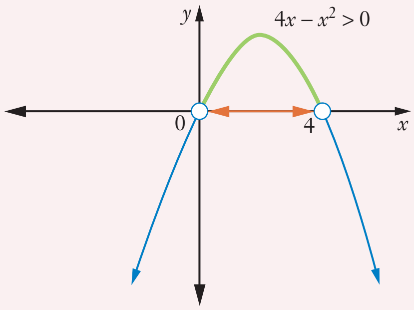
$0 = 4x - x^2$

$= x(4 - x)$

$x = 0, x = 4$

$4x - x^2 > 0$ above the x-axis.

$\therefore\ 0 < x < 4$



EXT1 Exercise 4.10 Quadratic inequalities

Solve each quadratic inequality.

1 $x^2 - 9 > 0$

2 $n^2 + n \leq 0$

3 $a^2 - 2a \geq 0$

4 $4 - x^2 < 0$

5 $y^2 - 6y \leq 0$

6 $2t - t^2 > 0$

7 $x^2 + 2x - 8 > 0$

8 $p^2 + 4p + 3 \geq 0$

9 $m^2 - 6m + 8 > 0$

10 $6 - x - x^2 \leq 0$

11 $2h^2 - 7h + 6 < 0$

12 $x^2 - x - 20 \leq 0$

13 $35 + 9k - 2k^2 \geq 0$

14 $q^2 - 9q + 18 > 0$

15 $(x + 2)^2 \geq 0$

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16 $12 - n - n^2 \le 0$ **17** $x^2 - 2x < 15$ **18** $-t^2 \ge 4t - 12$

19 $3y^2 > 14y + 5$ **20** $(x - 3)(x + 1) \ge 5$

4.11 The discriminant

WS The discriminant

The solutions of an equation are also called the **roots** of the equation.

In the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression $b^2 - 4ac$ is called the **discriminant**.

It gives us information about the roots of the quadratic equation $ax^2 + bx + c = 0$.

EXAMPLE 28

Use the quadratic formula to find how many real roots each quadratic equation has.

a $x^2 + 5x - 3 = 0$ **b** $x^2 - x + 4 = 0$ **c** $x^2 - 2x + 1 = 0$

Solution

a
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times (-3)}}{2 \times 1} \\ &= \frac{-5 \pm \sqrt{25 + 12}}{2} \\ &= \frac{-5 \pm \sqrt{37}}{2} \end{aligned}$$

There are 2 real roots:

$$x = \frac{-5 + \sqrt{37}}{2}, \frac{-5 - \sqrt{37}}{2}$$

b
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times 4}}{2 \times 1} \\ &= \frac{1 \pm \sqrt{-15}}{2} \end{aligned}$$

There are no real roots since $\sqrt{-15}$ has no real value.

c
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{0}}{2} \\ &= 1 \end{aligned}$$

There are 2 real roots:

$x = 1, 1$

However, these are equal roots.

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The discriminant

The value of the **discriminant** $\Delta = b^2 - 4ac$ tells us information about the roots of the quadratic equation $ax^2 + bx + c = 0$.

When $\Delta \geq 0$, there are 2 real roots.

- If Δ is a perfect square, the roots are rational.
- If Δ is not a perfect square, the roots are irrational.

When $\Delta = 0$, there are 2 equal rational roots (or 1 rational root).

When $\Delta < 0$, there are no real roots.

EXAMPLE 29

a Show that the equation $2x^2 + x + 4 = 0$ has no real roots.

b Describe the roots of the equation:

i $2x^2 - 7x - 1 = 0$ **ii** $x^2 + 6x + 9 = 0$

c Find the values of k for which the quadratic equation $5x^2 - 2x + k = 0$ has real roots.

Solution

a
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= 1^2 - 4(2)(4)\\ &= -31\\ &< 0\end{aligned}$$

$\Delta < 0$, so the equation has no real roots.

b **i**
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-7)^2 - 4(2)(-1)\\ &= 57\\ &> 0\end{aligned}$$

$\Delta > 0$, so there are 2 real irrational roots.

Roots are irrational because 57 is not a perfect square.

ii
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (6)^2 - 4(1)(9)\\ &= 0\end{aligned}$$

$\Delta = 0$ so there are 2 real equal rational roots.

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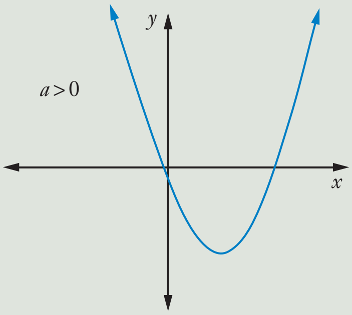
c For real roots, $\Delta \geq 0$.

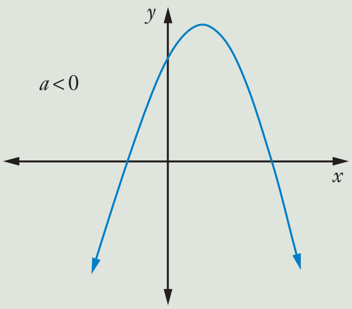
$$b^2 - 4ac \geq 0$$
$$(-2)^2 - 4(5)(k) \geq 0$$
$$4 - 20k \geq 0$$
$$4 \geq 20k$$
$$k \leq \frac{1}{5}$$

The discriminant and the parabola

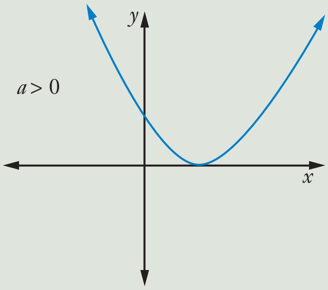
The roots of the quadratic equation $ax^2 + bx + c = 0$ give the x-intercepts of the parabola $y = ax^2 + bx + c$.

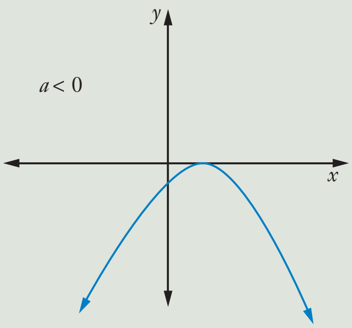
If $\Delta > 0$, then the quadratic equation has 2 real roots and the parabola has 2 x-intercepts.





If $\Delta = 0$, then the quadratic equation has 1 real root or 2 equal roots and the parabola has one x-intercept.





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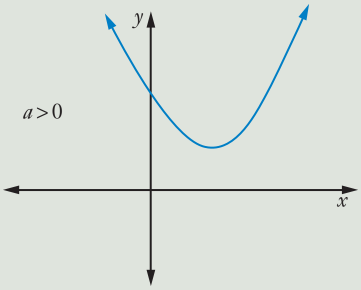
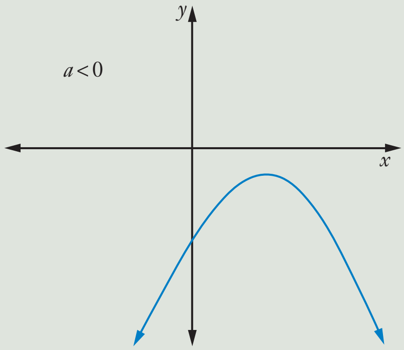
If $\Delta < 0$, then the quadratic equation has no real roots and the parabola has no x-intercepts.





If $\Delta < 0$ and $a > 0$, then $ax^2 + bx + c > 0$ for all x.

If $\Delta < 0$ and $a < 0$, then $ax^2 + bx + c < 0$ for all x.

EXAMPLE 30

a Show that the parabola $f(x) = x^2 - x - 2$ has 2 x-intercepts.

b Show that $x^2 - 2x + 4 > 0$ for all x.

Solution

a $\Delta = b^2 - 4ac$

$= (-1)^2 - 4(1)(-2)$

$= 9$

> 0

So there are 2 real roots and the parabola has 2 x-intercepts.

b If $a > 0$ and $\Delta < 0$, then $ax^2 + bx + c > 0$ for all x.

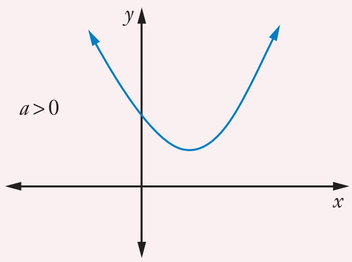
$a = 1 > 0$

$\Delta = b^2 - 4ac$

$= (-2)^2 - 4(1)(4)$

$= -12$

< 0



Since $a > 0$ and $\Delta < 0$, $x^2 - 2x + 4 > 0$ for all x.

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Exercise 4.11 The discriminant

1 Find the discriminant of each quadratic equation.

a $x^2 - 4x - 1 = 0$ **b** $2x^2 + 3x + 7 = 0$ **c** $-4x^2 + 2x - 1 = 0$

d $6x^2 - x - 2 = 0$ **e** $-x^2 - 3x = 0$ **f** $x^2 + 4 = 0$

g $x^2 - 2x + 1 = 0$ **h** $-3x^2 - 2x + 5 = 0$ **i** $-2x^2 + x + 2 = 0$

2 Find the discriminant and state whether the roots of the quadratic equation are real or not real. If the roots are real, state whether they are equal or unequal, rational or irrational.

a $x^2 - x - 4 = 0$ **b** $2x^2 + 3x + 6 = 0$ **c** $x^2 - 9x + 20 = 0$

d $x^2 + 6x + 9 = 0$ **e** $2x^2 - 5x - 1 = 0$ **f** $-x^2 + 2x - 5 = 0$

g $-2x^2 - 5x + 3 = 0$ **h** $-5x^2 + 2x - 6 = 0$ **i** $-x^2 + x = 0$

3 Find the value of p for which the quadratic equation $x^2 + 2x + p = 0$ has equal roots.

4 Find any values of k for which the quadratic equation $x^2 + kx + 1 = 0$ has equal roots.

5 Find all the values of b for which $2x^2 + x + b + 1 = 0$ has real roots.

6 Evaluate p if $px^2 + 4x + 2 = 0$ has no real roots.

7 Find all values of k for which $(k + 2)x^2 + x - 3 = 0$ has 2 real unequal roots.

8 Prove that $3x^2 - x + 7 > 0$ for all real x.

9 Show that the line $y = 2x + 6$ cuts the parabola $y = x^2 + 3$ in 2 points.

10 Show that the line $3x + y - 4 = 0$ cuts the parabola $y = x^2 + 5x + 3$ in 2 points.

11 Show that the line $y = -x - 4$ does not touch the parabola $y = x^2$.

12 Show that the line $y = 5x - 2$ is a tangent to the parabola $y = x^2 + 3x - 1$.

13 EXT1 Find the values of k for which $x^2 + (k + 1)x + 4 = 0$ has real roots.

14 EXT1 Find values of k for which the expression $kx^2 + 3kx + 9 > 0$ for all real x.

15 EXT1 Find the values of m for which the quadratic equation $x^2 - 2mx + 9 = 0$ has real and different roots.

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4.12 Finding a quadratic equation

A page of parabolas

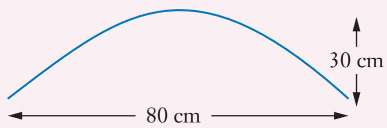
EXAMPLE 31

a Find the equation of the parabola that passes through the points $(-1, -3)$, $(0, 3)$ and $(2, 21)$.

b A parabolic satellite dish is built so it is 30 cm deep and 80 cm wide, as shown.

  i Find an equation for the parabola.

  ii Find the depth of the dish 10 cm out from the vertex.



Solution

a The parabola has equation in the form $y = ax^2 + bx + c$.

Substitute the points into the equation.

$(-1, -3)$:

$$-3 = a(-1)^2 + b(-1) + c$$
$$= a - b + c$$
$$\therefore\ a - b + c = -3 \qquad [1]$$

$(0, 3)$:

$$3 = a(0)^2 + b(0) + c$$
$$= c$$
$$\therefore\ c = 3 \qquad [2]$$

$(2, 21)$:

$$21 = a(2)^2 + b(2) + c$$
$$= 4a + 2b + c$$
$$\therefore\ 4a + 2b + c = 21 \qquad [3]$$

Solve simultaneous equations to find a, b and c.

Substitute [2] into [1]:

$$a - b + 3 = -3$$
$$a - b = -6 \qquad [4]$$

Substitute [2] into [3]:

$$4a + 2b + 3 = 21$$
$$4a + 2b = 18 \qquad [5]$$

[4] × 2:

$$2a - 2b = -12 \qquad [6]$$

[5] + [6]:

$$6a = 6$$
$$a = 1$$

Substitute $a = 1$ into [5]:

$$4(1) + 2b = 18$$
$$4 + 2b = 18$$
$$2b = 14$$
$$b = 7$$

$$\therefore\ a = 1, b = 7, c = 3$$

Thus the parabola has equation $y = x^2 + 7x + 3$.

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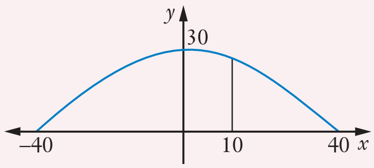
b **i** We can put the dish onto a number plane as shown. Since the parabola is symmetrical, the width of 80 cm means 40 cm either side of the y-axis.



The parabola passes through points (0, 30), (40, 0) and (–40, 0).

Substitute these points into $y = ax^2 + bx + c$.

(0, 30): $30 = a(0)^2 + b(0) + c = c$

So $y = ax^2 + bx + 30$

Substitute (40, 0) into $y = ax^2 + bx + 30$:

$0 = a(40)^2 + b(40) + 30$

$0 = 1600a + 40b + 30$ [1]

Substitute (–40, 0) into $y = ax^2 + bx + 30$:

$0 = a(-40)^2 + b(-40) + 30$

$0 = 1600a - 40b + 30$ [2]

[1] + [2]:

$$\begin{aligned} 0 &= 3200a + 60 \\ a &= \frac{-60}{3200} \\ &= -\frac{3}{160} \end{aligned}$$

Substitute a into [1]:

$$\begin{aligned} 0 &= 1600\left(-\frac{3}{160}\right) + 40b + 30 \\ &= -30 + 40b + 30 \\ &= 40b \\ 0 &= b \end{aligned}$$

So $y = -\frac{3}{160}x^2 + 30$

ii Substitute $x = 10$:

$$\begin{aligned} y &= -\frac{3}{160}(10)^2 + 30 \\ &= 28.125 \end{aligned}$$

So the depth of the dish at 10 cm is 28.125 cm.

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Exercise 4.12 Finding a quadratic equation

1 The braking distance of a car travelling at 100 km/h is 40 metres. The formula for braking distance (d) in metres is $d = kx^2$ where k is a constant and x is speed in km/h.

a Find the value of k.

b Find the braking distance at 80 km/h.

c A dog runs out onto the road 15 m in front of a car travelling at 50 km/h. Will the car be able to stop in time without hitting the dog?

d If the dog was 40 m in front of a car travelling at 110 km/h, would the car stop in time?

2 The area (A) of a figure is directly proportional to the square of its length (x). When $x = 5$ cm, its area is 125 cm^2.

a Find the equation for the area.

b Find the area when the length is 4.2 cm.

c Find the length correct to 1 decimal place when the area is 250 cm^2.

3 The volume of a cylinder is given by $V = \pi r^2 h$ where r is the radius and h is height.

a Find the equation for volume if the height is fixed at 8 cm.

b Find the volume of a cylinder with radius 5 cm.

c Find the radius if the volume is 100 cm^3.

4 A rectangle has sides x and $3 - x$.

a Write an equation for its area.

b Draw the graph of the area.

c Find the value of x that gives the maximum area.

d Find the maximum area of the rectangle.

5 Find the equation of the parabola that passes through the points:

a $(0, -5)$, $(2, -3)$ and $(-3, 7)$

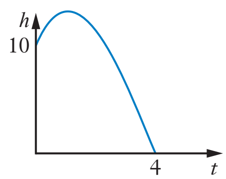
b $(1, -2)$, $(3, 0)$ and $(-2, 10)$

c $(-2, 21)$, $(1, 6)$ and $(-1, 12)$

d $(2, 3)$, $(1, -4)$ and $(-1, -12)$

e $(0, 1)$, $(-2, 1)$ and $(2, -7)$

6 Grania throws a ball off a 10 m high cliff. After 1 s it is 22.5 m above ground and it reaches the ground after 4 s.



a Find the equation for the height (h metres) of the ball after time t seconds.

b Find the height of the ball after 2 seconds.

c Find when the ball is in line with the cliff.

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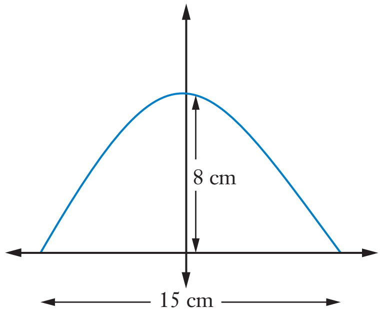
7 A parabolic shaped headlight is 15 cm wide and 8 cm deep as shown.

a Find an equation for the parabola.

b Find the depth of the headlight at a point 3 cm out from its axis of symmetry.

c At what width from the axis of symmetry does the headlight have a depth of 5 cm?



8 a Find the equation of the parabola passing through (0, 0), (3, –3) and (–1, 5).

b Find the value of y when:

i $x = 5$ **ii** $x = -4$

c Find values of x when $y = -4$.

d Find exact values of x when $y = 2$.

9 a Find the equation of the quadratic function $f(x)$ that passes through points (1, 10), (0, 7) and (–1, 6).

b Evaluate $f(-5)$.

c Show that $f(x) > 0$ for all x.

10 Find the equation of a parabola with axis of symmetry $x = 1$, minimum value –2 and passing through (0, 0).

11 Find the equation of the quadratic function with axis $x = 3$, maximum value 13 and passing through (0, 4).

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Cubic functions

Graphing cubics

Graphing cubics 2

4.13 Cubic functions

A **cubic function** has an equation where the highest power of x is 3, such as $f(x) = kx^3$, $f(x) = k(x - b)^3 + c$ and $f(x) = k(x - a)(x - b)(x - c)$ where a, b, c and k are constants.

EXAMPLE 32

a Sketch the graph of the cubic function $f(x) = x^3 + 2$.

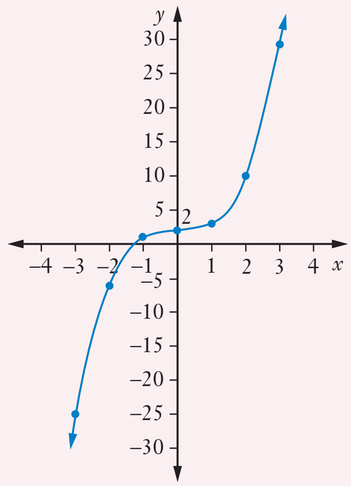
b State its domain and range.

c Solve the equation $x^3 + 2 = 0$ graphically.

Solution

a Draw up a table of values.

x	−3	−2	−1	0	1	2	3
y	−25	−6	1	2	3	10	29



b The function can have any real x or y value.

Domain $(-\infty, \infty)$

Range $(-\infty, \infty)$

c From the graph, the x-intercept is approximately -1.3.

So the root of $x^3 + 2 = 0$ is approximately $x = -1.3$.

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INVESTIGATION

TRANSFORMING CUBIC FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of some cubic functions, such as:

$y = x^3$	$y = x^3 + 1$	$y = x^3 + 3$	$y = x^3 - 1$
$y = x^3 - 2$	$y = 2x^3$	$y = 3x^3$	$y = -x^3$
$y = -2x^3$	$y = -3x^3$	$y = 2x^3 + 1$	$y = (x + 1)^3$
$y = (x + 2)^3$	$y = (x - 1)^3$	$y = 2(x - 2)^3$	$y = 3(x + 2)^3 + 1$

$y = (x - 1)(x - 2)(x - 3)$ $\quad$ $y = x(x + 1)(x + 4)$ $\quad$ $y = 2(x + 1)(x - 2)(x + 5)$

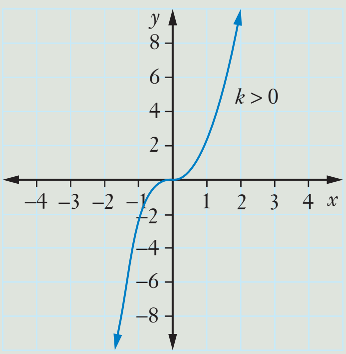
Can you see any patterns? Could you describe the shape of the cubic function? Could you predict where the graphs of different cubic functions would lie?

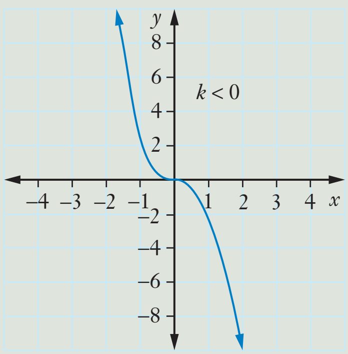
Is the cubic graph always a function? Can you find an example of a cubic that is not a function?

Point of inflection

The flat turning point of the cubic function $y = kx^3$ is called a **point of inflection**, which is where the concavity of the curve changes.

The graph of $y = kx^3$





This cubic curve is increasing and has a point of inflection at (0, 0) where the curve changes from concave downwards to concave upwards.

This cubic curve is decreasing and has a point of inflection at (0, 0) where the curve changes from concave upwards to concave downwards.

The graph of $y = k(x - b)^3 + c$

The graph of $y = k(x - b)^3 + c$ is the graph of $y = kx^3$ shifted so that its point of inflection is at (b, c).

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EXAMPLE 33

a Sketch the graph of $y = x^3 - 8$, showing intercepts.

b Sketch the graph of $f(x) = -2(x - 3)^3 + 2$.

Solution

a This is the graph of $y = x^3$ shifted downwards 8 units so that its point of inflection is at $(0, -8)$. Since $k > 0$, the function is increasing.

For x-intercepts, $y = 0$:

$0 = x^3 - 8$

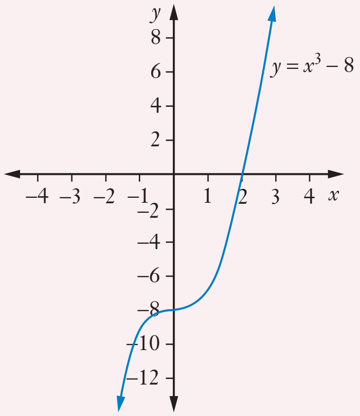
$8 = x^3$

$x = 2$

For y-intercept, $x = 0$:

$y = 0^3 - 8$

$= -8$



The point of inflection is at $(0, -8)$, where the curve changes from concave downwards to concave upwards.

b Since $k < 0$, $f(x)$ is decreasing.

This is the graph of $y = -2x^3$ shifted upwards and to the right so that its point of inflection is at $(3, 2)$.

For x-intercepts, $f(x) = 0$:

$0 = -2(x - 3)^3 + 2$

$-2 = -2(x - 3)^3$

$1 = (x - 3)^3$

$1 = x - 3$

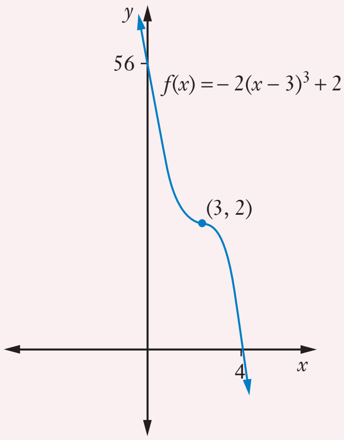
$x = 4$

For y-intercept, $x = 0$:

$y = -2(0 - 3)^3 + 2$

$= -2(-27) + 2$

$= 56$



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EXAMPLE 34

Show that $y = 2x^3$ is an odd function.

Solution

Let $f(x) = 2x^3$.

$$\begin{aligned} f(-x) &= 2(-x)^3 \\ &= -2x^3 \\ &= -f(x) \end{aligned}$$

So $y = 2x^3$ is an odd function.

A cubic function has one y-intercept and up to 3 x-intercepts. We can sketch the graph of a more general cubic function using intercepts. This will not give a very accurate graph but it will show the shape and important features.

The graph of $y = k(x - a)(x - b)(x - c)$

The graph of $y = k(x - a)(x - b)(x - c)$ has x-intercepts at a, b and c.

EXAMPLE 35

a **i** Sketch the graph of the cubic function $f(x) = x(x + 3)(x - 2)$.

ii Describe the shape of the graph and state its domain and range.

b Sketch the graph of the cubic function $f(x) = (x - 3)(x + 1)^2$ and describe its shape.

Solution

a **i** For x-intercepts, $f(x) = 0$:

$0 = x(x + 3)(x - 2)$

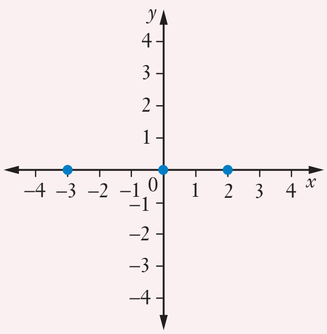
$x = 0, -3, 2$

Plot x-intercepts on graph.

For y-intercept, $x = 0$:

$$\begin{aligned} f(0) &= 0(0 + 3)(0 - 2) \\ &= 0 \end{aligned}$$

So y-intercept is 0.



We look at which parts of the graph are above and which are below the x-axis between the x-intercepts.

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Test $x < -3$, say $x = -4$:

$f(-4) = -4(-4 + 3)(-4 - 2) = -24 < 0$

So here the curve is below the x-axis.

Test $-3 < x < 0$, say $x = -1$:

$f(-1) = -1(-1 + 3)(-1 - 2) = 6 > 0$

So here the curve is above the x-axis.

Test $0 < x < 2$, say $x = 1$:

$f(1) = 1(1 + 3)(1 - 2) = -4 < 0$

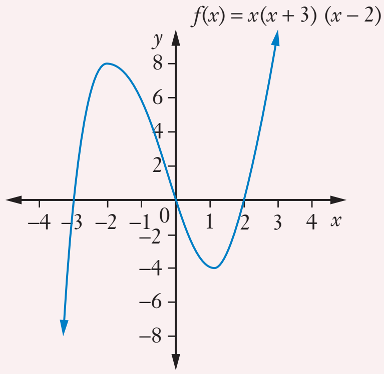
So here the curve is below the x-axis.

Test $x > 2$, say $x = 3$:

$f(3) = 3(3 + 3)(3 - 2) = 18 > 0$

So here the curve is above the x-axis.

We can sketch the cubic curve as shown.



ii The graph increases to a maximum turning point, then decreases to a minimum turning point. Then it increases again.

Domain $(-\infty, \infty)$

Range $(-\infty, \infty)$

b For x-intercepts, $f(x) = 0$:

$0 = (x - 3)(x + 1)^2$

$x = 3, x = -1$

So x-intercepts are -1 and 3.

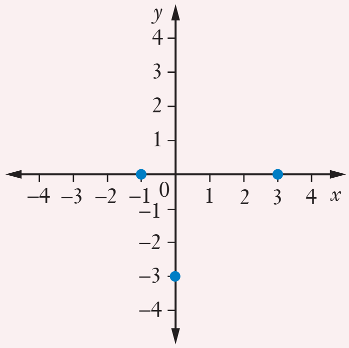
For y-intercept, $x = 0$:

$f(0) = (0 - 3)(0 + 1)^2$

$\quad = (-3)(1)$

$\quad = -3$

So y-intercept is -3.



We look at which parts of the graph are above and below the x-axis.

Test $x < -1$, say $x = -2$:

$f(-2) = (-2 - 3)(-2 + 1)^2 = -5 < 0$

So here the curve is below the x-axis.

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Test $-1 < x < 3$, say $x = 0$:

$f(0) = (0 - 3)(0 + 1)^2 = -3 < 0$

So here the curve is below the x-axis.

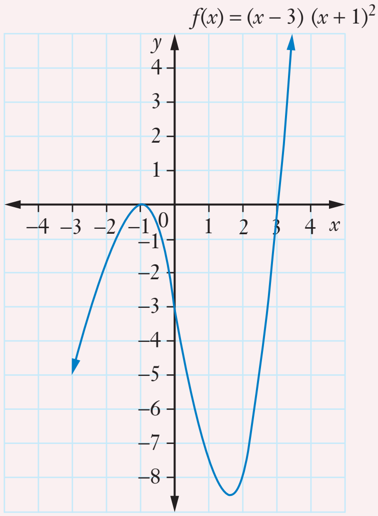
Test $x > 3$, say $x = 4$:

$f(4) = (4 - 3)(4 + 1)^2 = 25 > 0$

So here the curve is above the x-axis.

We can sketch the cubic curve as shown.

The graph increases to a maximum turning point, then decreases to a minimum turning point, then increases.



Finding a cubic equation

EXAMPLE 36

a Find the equation of the cubic function $y = kx^3 + c$ if it passes through (0, 16) and (4, 0).

b Find the equation of the cubic function $f(x) = k(x - a)(x - b)(x - c)$ if it has x-intercepts -1, 3 and 4 and passes through (1, 12).

Solution

a Substitute (0, 16) into $y = kx^3 + c$.

$16 = k(0)^3 + c$

$= c$

So $y = kx^3 + 16$.

Substitute (4, 0) into $y = kx^3 + 16$.

$0 = k(4)^3 + 16$

$= 64k + 16$

$-16 = 64k$

$k = -\frac{16}{64}$

$= -\frac{1}{4}$

So the equation is $y = -\frac{1}{4}x^3 + 16$.

b $f(x) = k(x - a)(x - b)(x - c)$ has x-intercepts when $f(x) = 0$.

$0 = k(x - a)(x - b)(x - c)$

$x = a, b, c$

But we know x-intercepts are at -1, 3 and 4.

So $a = -1$, $b = 3$ and $c = 4$ (in any order).

So $f(x) = k(x - (-1))(x - 3)(x - 4)$

$\quad = k(x + 1)(x - 3)(x - 4)$

To find k, substitute (1, 12):

$12 = k(1 + 1)(1 - 3)(1 - 4)$

$\quad = k(2)(-2)(-3)$

$\quad = 12k$

$1 = k$

So the cubic function is $f(x) = (x + 1)(x - 3)(x - 4)$.

Exercise 4.13 Cubic functions

1 Find the x- and y-intercept(s) of the graph of each cubic function.

a $y = x^3 - 1$ **b** $f(x) = -x^3 + 8$ **c** $y = (x + 5)^3$

d $f(x) = -(x - 4)^3$ **e** $f(x) = 3(x + 7)^3 - 3$ **f** $y = (x - 2)(x - 1)(x + 5)$

2 Draw each graph on a number plane.

a $y = -x^3$ **b** $p(x) = 2x^3$ **c** $g(x) = x^3 + 1$

d $y = (x + 2)^3$ **e** $y = -(x - 3)^3 + 1$ **f** $f(x) = -x(x + 2)(x - 4)$

g $y = (x + 2)(x - 3)(x + 6)$ **h** $y = x^2(x - 2)$ **i** $f(x) = (x - 1)(x + 3)^2$

3 Find the point of inflection of the graph of each cubic function by sketching each graph.

a $y = 8x^3 + 1$ **b** $y = -x^3 + 27$ **c** $f(x) = (x + 2)^3$

d $y = 2(x - 1)^3 - 16$ **e** $f(x) = -(x + 1)^3 + 1$

4 Find the x-intercept of the graph of each cubic function correct to one decimal place.

a $y = 2x^3 - 5$ **b** $f(x) = (x - 1)^3 + 2$ **c** $f(x) = -3x^3 + 1$

d $y = 2(x + 3)^3 - 3$ **e** $y = -3(2x - 1)^3 + 2$

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5 Describe the shape of each cubic function.

a $y = x^3 - 64$ **b** $f(x) = -(x - 3)^3$ **c** $y = x(x + 2)(x + 4)$

d $f(x) = -2(x + 3)(x + 1)(x - 4)$ **e** $y = x(x + 5)^2$

6 Solve graphically:

a $x^3 - 5 = 0$ **b** $x^3 + 2 = 0$ **c** $2x^3 - 9 = 0$

d $3x^3 + 4 = 0$ **e** $(x - 1)^3 + 6 = 0$ **f** $x(x + 2)(x - 1) = 0$

7 The volume of a certain solid has equation $V = kx^3$ where x is the length of its side in cm.

a Find the equation if $V = 120$ when $x = 3.5$.

b Find the volume when $x = 6$.

c Find x when $V = 250$.

8 The volume of a solid is directly proportional to the cube of its radius.

a If radius $r = 12$ mm when the volume V is 7238 mm^3, find an equation for the volume.

b Find the volume if the radius is 2.5 mm.

c Find the radius if the volume is 7000 mm^3.

9 Show that $f(x) = -x^3$ is an odd function.

10 Determine whether each function is odd.

a $y = 3x^3$ **b** $y = (x + 1)^3$ **c** $f(x) = -2x^3 - 1$

d $y = -5x^3$ **e** $y = (x - 2)^3 + 3$

11 A cubic function is in the form $y = kx^3 + c$. Find its equation if it passes through:

a (0, 0) and (1, 2) **b** (0, 5) and (2, –3) **c** (1, –4) and (–2, 23)

d (1, –2) and (2, 33) **e** (2, –29) and (–3, 111)

12 A cubic function is in the form $y = k(x - a)(x - b)(x - c)$. Find its equation if:

a it has x-intercepts 2, 3 and –5 and passes through the point (–2, –120)

b it has x-intercepts –1, 4 and 6 and passes through the point (3, 96)

c it has x-intercepts 1 and 3, y-intercept –27 and $k = -3$.

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4.14 Polynomial functions

Graphing power functions

A **polynomial** is a function defined for all real x involving powers of x in the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

where n is a positive integer or zero and $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

We generally write polynomials from the highest power down to the lowest, for example $P(x) = x^2 - 5x + 4$. We have already studied some polynomial functions, as linear, quadratic and cubic functions are all polynomials.

Polynomial terminology

$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$ is called a **polynomial expression**.

$P(x)$ has **degree** n (where n is the highest power of x).

$a_n, a_{n-1}, a_{n-2}, \ldots a_2, a_1$ and a_0 are called **coefficients**.

$a_n x^n$ is called the **leading term**.

a_n is the **leading coefficient**.

a_0 is called the **constant term**.

If $a_n = 1$, $P(x)$ is called a **monic polynomial**.

EXAMPLE 37

a Which of the following are polynomial expressions?

A $4 - x + 3x^2$ **B** $3x^4 - x^2 + 5x - 1$ **C** $x^2 - 3x + x^{-1}$

b $P(x) = x^6 - 2x^4 + 3x^3 + x^2 - 7x - 3$.

- **i** Find the degree of $P(x)$.
- **ii** Is the polynomial monic?
- **iii** State the leading term.
- **iv** What is the constant term?
- **v** Find the coefficient of x^4.

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Solution

a **A** and **B** are polynomials but **C** is not, because it has a term of x^{-1} that is not a positive integer power of x.

b
- **i** Degree is 6 since x^6 is the highest power.
- **ii** Yes, the polynomial is monic because the coefficient of x^6 is 1.
- **iii** The leading term is x^6.
- **iv** The constant term is -3.
- **v** The coefficient of x^4 is -2.

Polynomial equations

$P(x) = 0$ is a **polynomial equation** of degree n.

The values of x that satisfy the equation are called the **roots** of the equation or the **zeros** of the polynomial $P(x)$.

EXAMPLE 38

a Find the zeros of the polynomial $P(x) = x^2 - 5x$.

b Show that the polynomial $p(x) = x^2 - x + 4$ has no real zeros.

Solution

a To find the zeros of the polynomial, solve $P(x) = 0$.

$$x^2 - 5x = 0$$

$$x(x - 5) = 0$$

$x = 0, 5$

So the zeros are 0, 5.

b Solve $p(x) = 0$.

$x^2 - x + 4 = 0$

The discriminant will show whether the polynomial has real zeros.

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(1)(4) \\ &= -15 \\ &< 0 \end{aligned}$$

So the polynomial has no real zeros.

Graphing polynomials

EXAMPLE 39

a Write the polynomial $P(x) = x^4 + 2x^3 - 3x^2$ as a product of its factors.

b Sketch the graph of the polynomial.

Solution

a $P(x) = x^4 + 2x^3 - 3x^2$

$= x^2(x^2 + 2x - 3)$

$= x^2(x + 3)(x - 1)$

b For x-intercepts, $P(x) = 0$:

$0 = x^4 + 2x^3 - 3x^2$

$= x^2(x + 3)(x - 1)$

$x = 0, -3, 1$

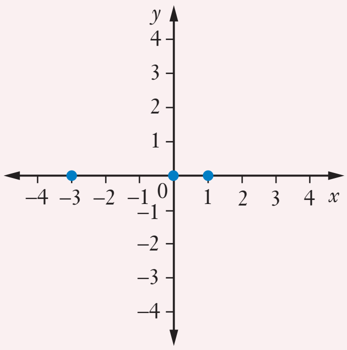
So the x-intercepts are $-3, 0, 1$.

For y-intercepts, $x = 0$:

$P(0) = 0^4 + 2(0)^3 - 3(0)^2$

$= 0$

So y-intercept is 0.



Test $x < -3$, say $x = -4$:

$P(-4) = (-4)^4 + 2(-4)^3 - 3(-4)^2 = 80 > 0$

So here the curve is above the x-axis.

Test $0 < x < 1$, say $x = 0.5$:

$P(0.5) = (0.5)^4 + 2(0.5)^3 - 3(0.5)^2$

$= -0.4375 < 0$

So here the curve is below the x-axis.

Test $-3 < x < 0$, say $x = -1$:

$P(-1) = (-1)^4 + 2(-1)^3 - 3(-1)^2 = -4 < 0$

So here the curve is below the x-axis.

Test $x > 1$, say $x = 3$:

$P(3) = 3^4 + 2(3)^3 - 3(3)^2 = 108 > 0$

So here the curve is above the x-axis.

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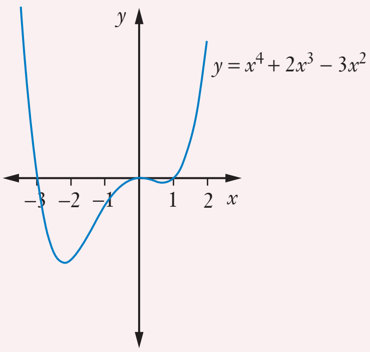
We can sketch the graph of the polynomial as shown.



DID YOU KNOW?

'Poly' means many

The word 'polynomial' means an expression with many terms. (A binomial has 2 terms and a trinomial has 3 terms.) 'Poly' means 'many', and is used in many words, for example polygamy, polyglot, polygon, polyhedron, polymer, polyphonic, polypod and polytechnic. Do you know what all these words mean? Do you know any others with 'poly-'?

Exercise 4.14 Polynomial functions

1 Write down the degree of each polynomial:

a $5x^7 - 3x^5 + 2x^3 - 3x + 1$ **b** $3 + x + x^2 - x^3 + 2x^4$ **c** $3x + 5$
d $x^{11} - 5x^8 + 4$ **e** $2 - x - 5x^2 + 3x^3$ **f** 3

2 For the polynomial $P(x) = x^3 - 7x^2 + x - 1$, find:

a $P(2)$ **b** $P(-1)$ **c** $P(0)$

3 Given $P(x) = x + 5$ and $Q(x) = 2x - 1$, find:

a $P(-11)$ **b** $Q(3)$ **c** $P(2) + Q(-2)$
d the degree of $P(x) + Q(x)$ **e** the degree of $P(x)Q(x)$

4 For the polynomial $P(x) = x^5 - 3x^4 - 5x + 4$, find:

a the degree of $P(x)$ **b** the constant term
c the coefficient of x^4 **d** the coefficient of x^2.

5 Find the zeros of each polynomial.

a $P(x) = x^2 - 9$ **b** $p(x) = x + 5$ **c** $f(x) = x^2 + x - 2$
d $P(x) = x^2 - 8x + 16$ **e** $g(x) = x^3 - 2x^2 + 5x$

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6 Which of the following are not polynomials?

a $5x^4 - 3x^2 + x + \frac{1}{x}$ **b** $x^2 + 3^x$ **c** $x^2 + 3x - 7$

d $3x + 5$ **e** 0 **f** $4x^3 + 7x^{-2} + 5$

7 For the polynomial $P(x) = (a + 1)x^3 + (b - 7)x^2 + c + 5$, find values for a, b or c if:

a $P(x)$ is monic **b** the coefficient of x^2 is 3

c the constant term is -1 **d** $P(x)$ has degree 2

e the leading term has a coefficient of 5.

8 Given $P(x) = 2x + 5$, $Q(x) = x^2 - x - 2$ and $R(x) = x^3 + 9x$, find:

a any zeros of $P(x)$ **b** the roots of $Q(x) = 0$

c the degree of $P(x) + R(x)$ **d** the degree of $P(x)Q(x)$

e the leading term of $Q(x)R(x)$.

9 Given $f(x) = 3x^2 - 2x + 1$ and $g(x) = 3x - 3$:

a show $f(x)$ has no zeros **b** find the leading term of $f(x)g(x)$

c find the constant term of $f(x) + g(x)$ **d** find the coefficient of x in $f(x)g(x)$

e find the roots of $f(x) + g(x) = 0$.

10 State how many real roots there are for each polynomial equation $P(x) = 0$.

a $P(x) = x^2 - 9$ **b** $P(x) = x^2 + 4$

c $P(x) = x^2 - 3x - 7$ **d** $P(x) = 2x^2 + x + 3$

e $P(x) = 3x^2 - 5x - 2$ **f** $P(x) = x(x - 1)(x + 4)(x + 6)$

11 Sketch the graph of each polynomial by finding its zeros and showing the x- and y-intercepts.

a $f(x) = (x + 1)(x - 2)(x - 3)$ **b** $P(x) = x\,(x + 4)(x - 2)$

c $p(x) = -x(x - 1)(x - 3)$ **d** $f(x) = x(x + 2)^2$

e $g(x) = (5 - x)(x + 2)(x + 5)$

12 **i** Write each polynomial as a product of its factors.

ii Sketch the graph of the polynomial and describe its shape.

a $P(x) = x^3 - 2x^2 - 8x$ **b** $f(x) = -x^3 - 4x^2 + 5x$

c $P(x) = x^4 + 3x^3 + 2x^2$ **d** $A(x) = 2x^3 + x^2 - 15x$

e $P(x) = -x^4 + 2x^3 + 3x^2$

13 **a** Find the x-intercepts of the polynomial $P(x) = x(x - 1)(x + 2)^2$.

b Sketch the graph of the polynomial.

14 **a** Show that $(x - 3)(x - 2)(x + 2) = x^3 - 3x^2 - 4x + 12$.

b Sketch the graph of the polynomial $P(x) = x^3 - 3x^2 - 4x + 12$.

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4.15 Intersection of graphs

Solving equations graphically

EXAMPLE 40

a Sketch $y = x^2$ and $y = 1$ on the same set of axes, and hence solve $x^2 = 1$ graphically.

b Sketch $y = x^2 - x$ and $y = 2$ on the same set of axes, and hence solve $x^2 - x = 2$ graphically.

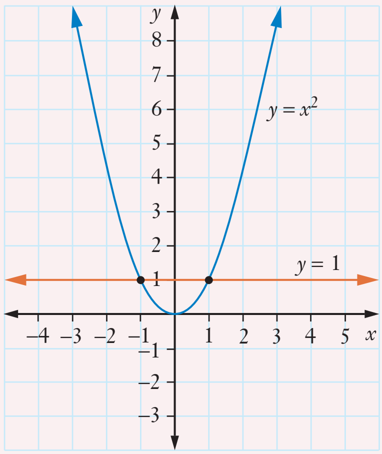
c EXT1 Solve $x^2 - x \leq 2$ graphically.

Solution

a $y = x^2$ is a parabola and $y = 1$ is a horizontal line, as shown.

To solve $x^2 = 1$ graphically, find the x values where the 2 graphs $y = x^2$ and $y = 1$ intersect.

The solution is $x = \pm 1$.

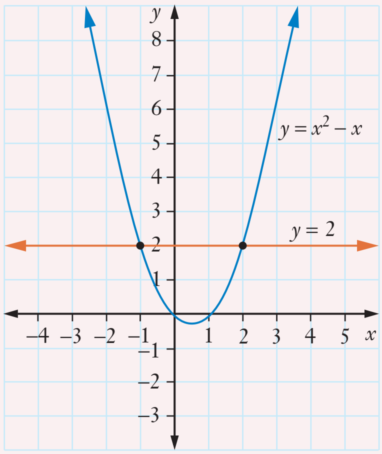


b $y = x^2 - x$ is a parabola with x-intercepts 0, 1 and y-intercept 0. Since $a > 0$, it is concave upwards.

$y = 2$ is a horizontal line.

The solutions of $x^2 - x = 2$ are the x values at the intersection of the 2 graphs.

$x = -1, 2$.



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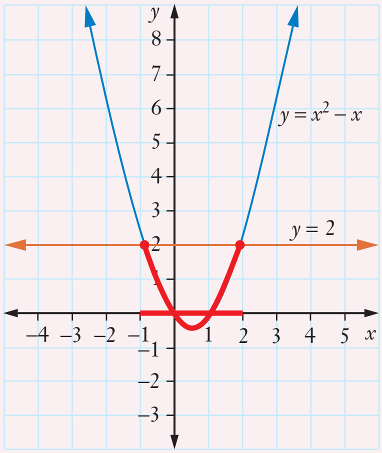
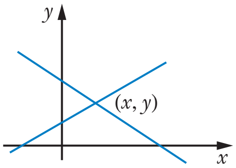
c The solutions of $x^2 - x \le 2$ are the x values at and below the intersection of the 2 graphs.

$-1 \le x \le 2$.



Intersecting lines

Two straight lines intersect at a single point (x, y).



The point of intersection can be found graphically or algebraically using simultaneous equations.

EXAMPLE 41

Find the point of intersection between lines $2x - 3y - 3 = 0$ and $5x - 2y - 13 = 0$.

Solution

Solve simultaneous equations.

$2x - 3y - 3 = 0 \qquad [1]$

$5x - 2y - 13 = 0 \qquad [2]$

[1] × 2:

$4x - 6y - 6 = 0 \qquad [3]$

[2] × 3:

$15x - 6y - 39 = 0 \qquad [4]$

[3] − [4]:

$$-11x + 33 = 0$$
$$33 = 11x$$
$$3 = x$$

Substitute $x = 3$ into [1]:

$$2(3) - 3y - 3 = 0$$
$$-3y + 3 = 0$$
$$3 = 3y$$
$$1 = y$$

So the point of intersection is (3, 1).

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Break-even points

Break-even points

EXAMPLE 42

A company that manufactures cables sells them for \$2 each. It costs 50 cents to produce each cable and the company has fixed costs of \$1500 per week.

a Find the equation for the income, \$$I$, on x cables per week.

b Find the equation for the costs, \$$C$, of manufacturing x cables per week.

c Find the break-even point (where income = costs).

d Find the profit on 1450 cables.

Solution

a $I = 2x$

b $C = 0.5x + 1500$

c Solving simultaneous equations:

$$I = 2x \quad [1]$$

$$C = 0.5x + 1500 \quad [2]$$

Substitute [1] into [2]:

$$2x = 0.5x + 1500$$

$$1.5x = 1500$$

$$x = 1000$$

1000 cables is where income = costs.
Substitute $x = 1000$ into [1] (or [2]):

$$I = 2(1000)$$

$$= 2000$$

So the break-even point is (1000, 2000). 1000 cables gives an income and cost of \$2000.

d Profit = income − costs = $I - C$

Substitute $x = 1450$ into both equations.

$$I = 2x$$

$$= 2(1450)$$

$$= 2900$$

So income is \$2900.

$$C = 0.5x + 1500$$

$$= 0.5(1450) + 1500$$

$$= 2225$$

So costs are \$2225.

Profit = \$2900 − \$2225

= \$675

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Break-even point

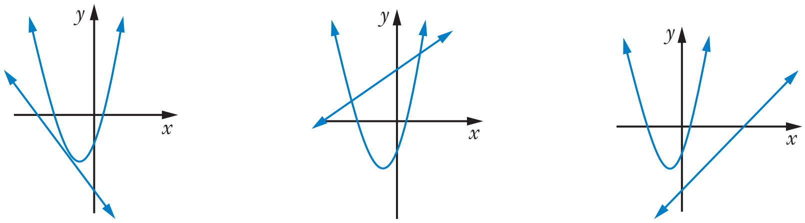
In business, the **break-even point** is the point where the **income** (or **revenue**) **equals costs.**

If income > costs, the business makes a **profit**.

If income < costs, the business makes a **loss**.

Intersecting lines and parabolas

A line and a parabola can intersect at 1 or 2 points, or they may not intersect at all.



EXAMPLE 43

Find the points of intersection of the line $y = x - 1$ with the parabola $y = x^2 + 4x + 1$.

Solution

Solve simultaneous equations.

$y = x - 1$ [1]

$y = x^2 + 4x + 1$ [2]

Substitute [1] into [2]:

$$x - 1 = x^2 + 4x + 1$$
$$0 = x^2 + 3x + 2$$
$$= (x + 2)(x + 1)$$
$$x = -2, -1$$

Substitute $x = -2$ into [1]:

$$y = -2 - 1$$
$$= -3$$

Substitute $x = -1$ into [1]:

$$y = -1 - 1$$
$$= -2$$

So the 2 points of intersection are $(-2, -3)$ and $(-1, -2)$.

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Exercise 4.15 Intersection of graphs

1 **a** Given $f(x) = 2x - 4$, solve graphically:

i $f(x) = 0$

ii $f(x) = -2$

iii $f(x) = 4$

b By sketching the graph of $f(x) = x^2 - 2x$, solve graphically:

i $f(x) = 0$

ii $f(x) = 3$

c Use the sketch of $f(x) = x^3 - 1$ to solve graphically:

i $f(x) = 0$

ii $f(x) = 7$

iii $f(x) = -2$

d EXT1 Solve graphically:

i $x^2 - 3x > 0$

ii $x^2 - 4 \le 0$

iii $x^2 - 7x > -12$

iv $m^2 < 4m$

2 Find the point of intersection between:

a $y = x + 3$ and $y = 2x + 2$

b $y = 3x - 1$ and $y = 5x + 1$

c $x + 2y - 4 = 0$ and $2x - y + 2 = 0$

d $3x + y - 2 = 0$ and $2x - 3y - 5 = 0$

e $4x - 3y - 5 = 0$ and $7x - 2y - 12 = 0$

3 Find points of intersection between:

a $y = x^2$ and $y = x$

b $y = x^2$ and $y = 4$

c $y = x^2$ and $y = x + 2$

d $y = x^2$ and $y = -2x + 3$

e $y = x^2 - 5$ and $y = 4x$

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4 **a** Draw the graphs of $f(x) = x^2$ and $f(x) = (x - 2)^2$ on the same number plane.

b From the graph, find the number of points of intersection of the functions.

c From the graph or by using algebra, find any points of intersection.

5 Find any points of intersection between the functions $f(x) = x^2$ and $f(x) = (x + 2)^2$.

6 Find any points of intersection between the curves $y = x^2 - 5$ and $y = 2x^2 + 5x + 1$.

7 Find any points of intersection between $y = 3x^2 - 4x - 4$ and $y = 5x^2 - 2$.

8 **a** If Paula's Posies' income on x roses is given by $y = 10x$ and the costs are $y = 3x + 980$, find the break-even point.

b Find the profit on 189 roses.

c Find the loss on 45 roses.

9 Find the number of calculators that a company needs to sell to break even each week if it costs \$3 to make each calculator and they are sold for \$15 each. Fixed overheads are \$852 a week.

10 Cupcakes Online sells cupcakes at \$5 each. The cost of making each cupcake is \$1 and the company has fixed overheads of \$264 a day.

a Find the equations for daily income and costs.

b Find how many cupcakes the company needs to sell daily to break even.

c What is the profit on 250 cupcakes?

d What is the loss on 50 cupcakes?

11 **a** The perimeter of a figure is in direct proportion to its side x. Find an equation for perimeter if the perimeter $y = 90$ cm when side $x = 5$ cm.

b The area of the figure is in direct proportion to the square of its side x. If the area of the figure is $y = 108 \text{ cm}^2$ when $x = 3$ cm, find its equation.

c Find any x values for the side for which the perimeter and area will have the same y value.

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4. TEST YOURSELF

For Questions **1** to **5**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 Which polynomial below is a monic polynomial with constant term 5 and degree 6?

A $P(x) = -x^6 + 5$
B $P(x) = 6x^5 - 3x^4 + 5$
C $P(x) = x^6 - 3x^4 + 5$
D $P(x) = 5x^6 - 3x^4 + 1$

2 The axis of symmetry and turning point of the quadratic function $f(x) = 1 + 2x - x^2$ are, respectively:

A $x = 1$, $(1, 2)$
B $x = -1$, $(-1, 4)$
C $x = 2$, $(2, 5)$
D $x = -2$, $(-2, 5)$

3 The linear function $2x - 3y - 6 = 0$ has x- and y-intercepts, respectively:

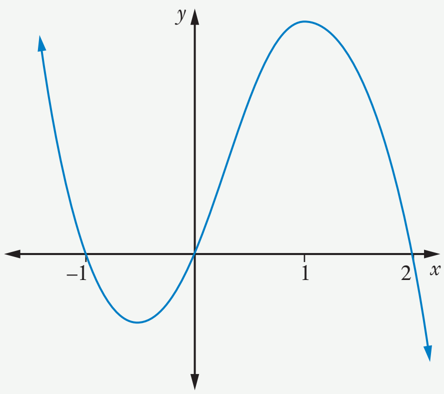
A -3 and 2
B 3 and -2
C -3 and -2
D 3 and 2

4 The domain and range of the straight line with equation $x = -2$ are:

A Domain $(-\infty, \infty)$, range $[-2]$
B Domain $[-2]$, range $(-\infty, \infty)$
C Domain $(-\infty, \infty)$, range $(-\infty, \infty)$
D Domain $[-2]$, range $[-2]$

5 Which cubic function has this graph?

A $y = x(x + 1)(x - 2)$
B $y = -x(x - 1)(x + 2)$
C $y = x(x - 1)(x + 2)$
D $y = -x(x + 1)(x - 2)$



6 If $f(x) = x^2 - 3x - 4$, find:

a $f(-2)$
b $f(a)$
c x when $f(x) = 0$

7 Sketch each graph and find its domain and range.

a $y = x^2 - 3x - 4$
b $f(x) = x^3$
c $2x - 5y + 10 = 0$
d $x = 3$
e $y = (x + 1)^3$
f $y = -2$
g $f(x) = -x^2 + x$
h $f(x) = x^2 + 4x + 4$

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8 If $f(x) = 3x - 4$, find:

a $f(2)$ **b** x when $f(x) = 7$ **c** x when $f(x) = 0$

9 Sketch the graph of $P(x) = 2x^3 - 2x^2 - 4x$.

10 Find the gradient of the straight line:

a passing through $(3, -1)$ and $(-2, 5)$ **b** with equation $2x - y + 1 = 0$

c perpendicular to the line $5x + 3y - 8 = 0$ **d** making an angle of inclination of 45°.

11 For the parabola $y = x^2 - 4x + 1$, find:

a the equation of the axis of symmetry

b the minimum value.

12 Sketch the graph of $f(x) = (x - 2)(x + 3)^2$, showing the intercepts.

13 For the polynomial $P(x) = x^3 + 2x^2 - 3x$, find:

a the degree **b** the coefficient of x

c the zeros **d** the leading term.

14 Find the x- and y-intercepts of:

a $2x - 5y + 20 = 0$ **b** $y = x^2 - 5x - 14$

c $y = (x + 2)^3$ **d** $2x - 5y - 10 = 0$

15 Find the point of intersection between lines $y = 2x + 3$ and $x - 5y + 6 = 0$.

16 For the quadratic function $y = -2x^2 - x + 6$, find:

a the equation of the axis of symmetry

b the maximum value.

17 Find the domain and range of $y = -2x^2 - x + 6$.

18 For each quadratic equation, select the correct property of its roots **A**, **B**, **C** or **D**.

A real, different and rational **B** real, different and irrational

C equal **D** unreal.

a $2x^2 - x + 3 = 0$ **b** $x^2 - 10x - 25 = 0$ **c** $x^2 - 10x + 25 = 0$

d $3x^2 + 7x - 2 = 0$ **e** $6x^2 - x - 2 = 0$

19 Find the equation of the line:

a passing through $(2, 3)$ and with gradient 7

b parallel to the line $5x + y - 3 = 0$ and passing through $(1, 1)$

c through the origin, and perpendicular to the line $2x - 3y + 6 = 0$

d through $(3, 1)$ and $(-2, 4)$

e with x-intercept 3 and y-intercept -1.

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20 The polynomial $f(x) = ax^2 + bx + c$ has zeros 4 and 5, and $f(-1) = 60$. Evaluate a, b and c.

21 Determine whether each function is even, odd or neither.

a $y = x^2 - 1$ **b** $y = x + 1$ **c** $y = x^3$

d $y = (x + 1)^2$ **e** $y = -5x^3$

22 Show that $f(x) = x^3 - x$ is odd.

23 Prove that the line between $(-1, 4)$ and $(3, 3)$ is perpendicular to the line $4x - y - 6 = 0$.

24 Show that $-4 + 3x - x^2 < 0$ for all x.

25 For each pair of equations, state whether their graphs have 0, 1 or 2 points of intersection.

a $xy = 7$ and $3x - 5y - 1 = 0$ **b** $x^2 + y^2 = 9$ and $y = 3x - 3$

c $x^2 + y^2 = 1$ and $x - 2y - 3 = 0$ **d** $y = x^2$ and $y = 4x - 4$

e EXT1 $y = \frac{2}{x}$ and $y = 3x + 1$

26 Prove that the lines with equations $y = 5x - 7$ and $10x - 2y + 1 = 0$ are parallel.

27 Find the zeros of $g(x) = -x^2 + 9x - 20$.

28 Sketch the graph of $P(x) = 2x(x - 3)(x + 5)$, showing intercepts.

29 Solve $P(x) = 0$ when $P(x) = x^3 - 4x^2 + 4x$.

30 Find x if the gradient of the line through $(3, -4)$ and $(x, 2)$ is -5.

31 If $f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x^2 - 3 & \text{if } x < 1 \end{cases}$, find $f(5) - f(0) + f(1)$.

32 Given $f(x) = \begin{cases} 3 & \text{if } x > 3 \\ x^2 & \text{if } 1 \leq x \leq 3 \\ 2 - x & \text{if } x < 1 \end{cases}$

find:

a $f(2)$ **b** $f(-3)$ **c** $f(3)$

d $f(5)$ **e** $f(0)$

33 Find the equation of the parabola:

a that passes through the points $(-2, 18)$, $(3, -2)$ and $(1, 0)$

b with x-intercepts 3 and -2 and y-intercept 12.

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34 The area (A) of a certain shape is in direct proportion to the square of its length x. If the area is 448 cm^2 when $x = 8$, find:

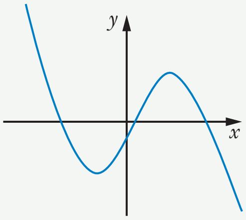
a the equation for area

b the area when $x = 10$

c x when the area is 1093.75 cm^2.

35 For each graph and set of ordered pairs, state whether it represents a function, and for those that do, whether it represents a one-to-one function.

a



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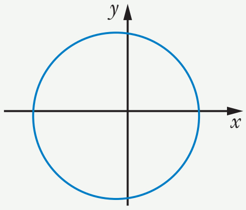

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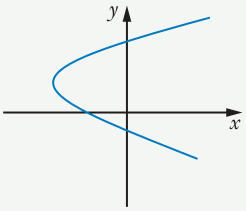
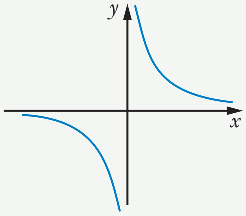


d



e (1, 2), (2, 5), (–1, 4), (1, 3), (3, 4)

36 Find the equation of a cubic function $f(x) = kx^3 + c$ if it passes through the point (1, 2) and has y-intercept 5.

37 A company has costs given by $y = 7x + 15$ and income $y = 12x$. Find the break-even point.

38 **a** Find the equation of the straight line that is perpendicular to the line $y = \frac{1}{2}x - 3$ and passes through (1, –1).

b Find the x-intercept of this line.

39 Find values of m such that $mx^2 + 3x - 4 < 0$ for all x.

40 Find any points of intersection of the graphs of:

a $y = 3x - 4$ and $y = 1 - 2x$

b $y = x^2 - x$ and $y = 2x - 2$

c $y = x^2$ and $y = 2x^2 - 9$

41 Find the equation of the straight line passing through the origin and parallel to the line with equation $3x - 4y + 5 = 0$.

42 Find the equation of the line with y-intercept –2 and perpendicular to the line passing through (3, –2) and (0, 5).

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43 The amount of petrol used in a car is directly proportional to the distance travelled.

a If the car uses 10.8 litres of petrol for an 87 km trip, find the equation for the amount of petrol used (A) over a distance of d km.

b Find the amount of petrol used for a 250 km trip.

c Find how far the car travelled if it used 35.5 L of petrol.

44 EXT1 Solve each inequality.

a $x^2 - 3x \leq 0$ **b** $n^2 - 9 > 0$ **c** $4 - y^2 \geq 0$

45 A function has equation $f(x) = x^3 - x^2 - 4x + 4$.

a Solve $f(x) = 0$.

b Find its x- and y-intercepts.

c Sketch the graph of the function.

d From the graph, state how many solutions there are for:

i $f(x) = 1$

ii $f(x) = -2$

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4. CHALLENGE EXERCISE

1 Find the values of b if $f(x) = 3x^2 - 7x + 1$ and $f(b) = 7$.

2 Sketch the graph of $y = (x + 2)^2 - 1$ in the domain $[-3, 0]$.

3 If points $(-3k, 1)$, $(k - 1, k - 3)$ and $(k - 4, k - 5)$ are collinear (lie on a straight line), find the value of k.

4 Find the equation of the line that passes through the point of intersection of the lines $2x + 5y + 19 = 0$ and $4x - 3y - 1 = 0$ and is perpendicular to the line $3x - 2y + 1 = 0$.

5 If $ax - y - 2 = 0$ and $bx - 5y + 11 = 0$ intersect at the point $(3, 4)$, find the values of a and b.

6 By writing each as a quadratic equation, solve:

a $(3x - 2)^2 - 2(3x - 2) - 3 = 0$

b $5^{2x} - 26(5^x) + 25 = 0$

c $2^{2x} - 10(2^x) + 16 = 0$

d $2^{2x+1} - 5(2^x) + 2 = 0$

e EXT1 $\left(x+\dfrac{1}{x}\right)^2 - 5\left(x+\dfrac{1}{x}\right) + 6 = 0$

7 Find the equation of the straight line through $(1, 3)$ that passes through the intersection of the lines $2x - y + 5 = 0$ and $x + 2y - 5 = 0$.

8 $f(x) = \begin{cases} 2x+3 & \text{when } x > 2 \\ 1 & \text{when } -2 \le x \le 2 \\ x^2 & \text{when } x < -2 \end{cases}$

Find $f(3)$, $f(-4)$, $f(0)$ and sketch the graph of the function.

9 If $h(t) = \begin{cases} 1-t^2 & \text{if } t > 1 \\ t^2-1 & \text{if } t \le 1 \end{cases}$, find the value of $h(2) + h(-1) - h(0)$ and sketch the curve.

10 If $f(x) = 2x^3 - 2x^2 - 12x$, find x when $f(x) = 0$.

11 Show that the quadratic equation $2x^2 - kx + k - 2 = 0$ has real rational roots.

12 Find the values of p for which $x^2 - x + 3p - 2 > 0$ for all x.

13 If $f(x) = 2x - 1$ show that $f(a^2) = f[(-a)^2]$ for all real a.

14 Find the equation of the straight line through $(3, -4)$ that is perpendicular to the line with x-intercept -2 and y-intercept 5.

15 Find any points of intersection between $y = x^2$ and $y = x^3$.

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16 Find the equation of a cubic function $y = ax^3 + bx^2 + cx + d$ if it passes through (0, 1), (1, 3), (–1, 3) and (2, 15).

17 Show that the quadratic equation $x^2 - 2px + p^2 = 0$ has equal roots.

18 EXT1 Solve $x^2 + 1 + \dfrac{25}{x^2 + 1} = 10$.

19 EXT1 Find exact values of k for which $x^2 + 2kx + k + 5 = 0$ has real roots.

20 A monic polynomial $P(x)$ of degree 3 has zeros –2, 1 and 6. Write down the equation of the polynomial.