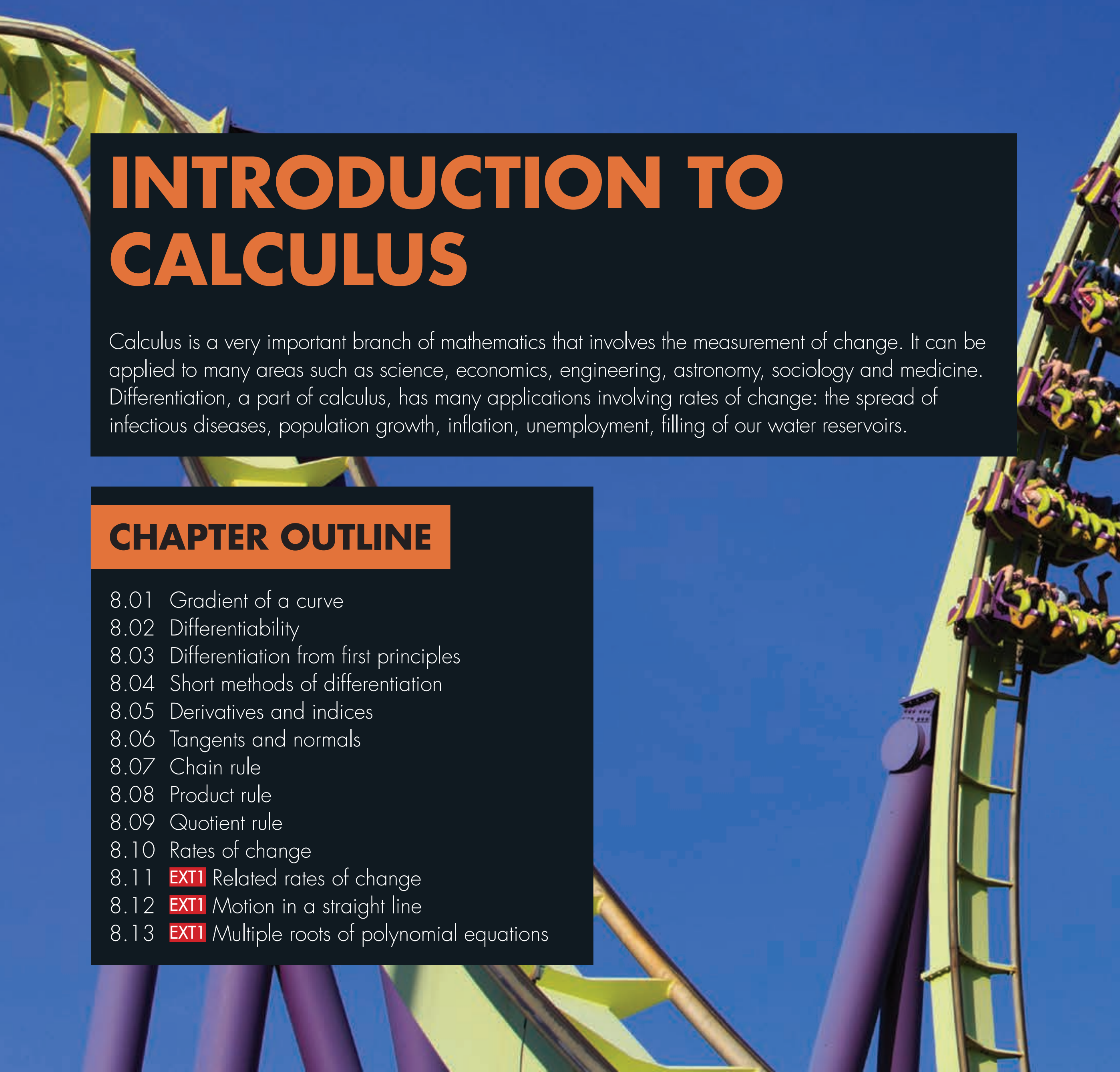
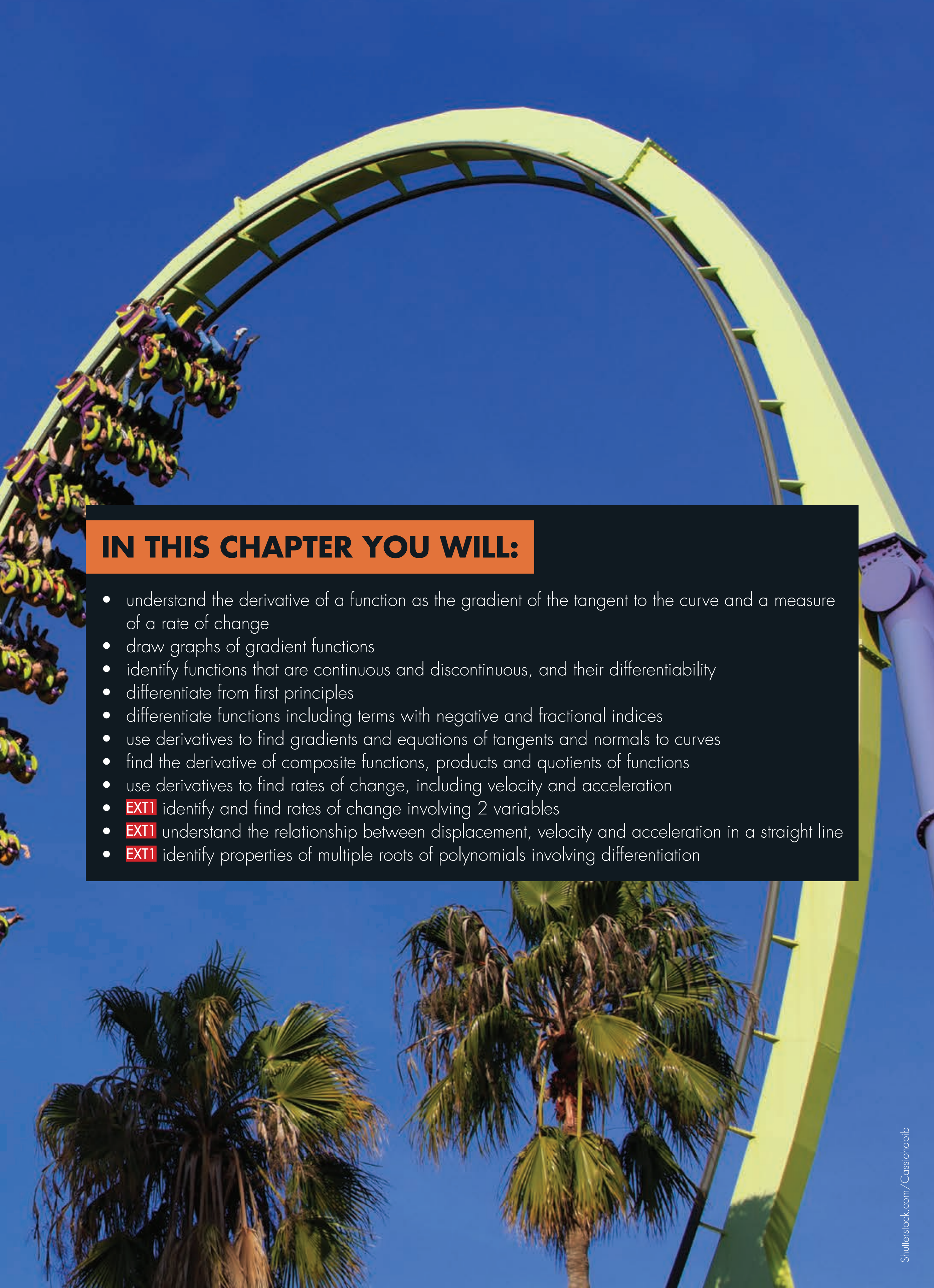


8.

INTRODUCTION TO CALCULUS

Calculus is a very important branch of mathematics that involves the measurement of change. It can be applied to many areas such as science, economics, engineering, astronomy, sociology and medicine. Differentiation, a part of calculus, has many applications involving rates of change: the spread of infectious diseases, population growth, inflation, unemployment, filling of our water reservoirs.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand the derivative of a function as the gradient of the tangent to the curve and a measure of a rate of change
- draw graphs of gradient functions
- identify functions that are continuous and discontinuous, and their differentiability
- differentiate from first principles
- differentiate functions including terms with negative and fractional indices
- use derivatives to find gradients and equations of tangents and normals to curves
- find the derivative of composite functions, products and quotients of functions
- use derivatives to find rates of change, including velocity and acceleration
- EXT1 identify and find rates of change involving 2 variables
- EXT1 understand the relationship between displacement, velocity and acceleration in a straight line
- EXT1 identify properties of multiple roots of polynomials involving differentiation

TERMINOLOGY

acceleration: The rate of change of velocity with respect to time

average rate of change: The rate of change between 2 points on a function; the gradient of the line (secant) passing through those points

chain rule: A method for differentiating composite functions

derivative function: The gradient function $y = f'(x)$ of a function $y = f(x)$ obtained through differentiation

differentiability: A function is differentiable wherever its gradient is defined

differentiation: The process of finding the gradient function

differentiation from first principles: The process of finding the gradient of a tangent to a curve by finding the gradient of the secant between 2 points and finding the limit as the secant becomes a tangent

displacement: The distance and direction of an object in relation to the origin

gradient of a secant: The gradient (slope) of the line between 2 points on a function; measures the average rate of change between the 2 points

gradient of a tangent: The gradient of a line that is a tangent to the curve at a point on a function; measures the instantaneous rate of change of the function at that point

instantaneous rate of change: The rate of change at a particular point on a function; the gradient of the tangent at this point

limit: The value that a function approaches as the independent variable approaches some value

normal: A line that is perpendicular to the tangent at a given point on a curve

product rule: A method for differentiating the product of 2 functions

quotient rule: A method for differentiating the quotient of 2 functions

secant: A straight line passing through 2 points on the graph of a function

stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection

tangent: A straight line that just touches a curve at one point. The curve has the same gradient or direction as the tangent at that point

turning point: A maximum or minimum point on a curve, where the curve turns around

velocity: The rate of change of displacement of an object with respect to time; involves speed and direction

DID YOU KNOW?

Newton and Leibniz

'Calculus' comes from the Latin meaning 'pebble' or 'small stone'. In many ancient civilisations stones were used for counting, but the mathematics they practised was quite sophisticated.

It was not until the 17th century that there was a breakthrough in calculus when scientists were searching for ways of measuring motion of objects such as planets, pendulums and projectiles.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Isaac Newton

Isaac Newton (1642–1727), an Englishman, discovered the main principles of calculus when he was 23 years old. At this time an epidemic of bubonic plague had closed Cambridge University where he was studying, so many of his discoveries were made at home. He first wrote about his calculus methods, which he called fluxions, in 1671, but his *Method of fluxions* was not published until 1704.

Gottfried Leibniz (1646–1716), in Germany, was studying the same methods and there was intense rivalry between the two countries over who was first to discover calculus!

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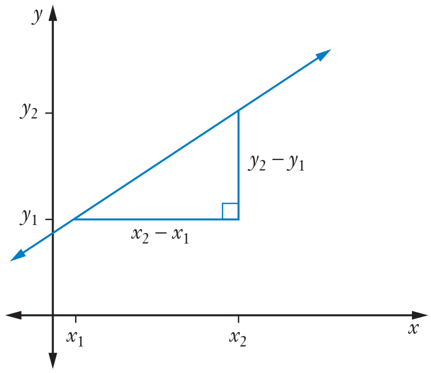
8.01 Gradient of a curve

WS Gradient functions
WS Gradient functions

The **gradient** of a straight line measures the **rate of change** of y (the dependent variable) with respect to the change in x (the independent variable).



Gradient

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Notice that when the gradient of a straight line is positive the line is increasing, and when the gradient is negative the line is decreasing. Straight lines increase or decrease at a constant rate and the gradient is the same everywhere along the line.

CLASS DISCUSSION

Remember that an **increasing** line has a positive gradient and a **decreasing** line has a negative gradient.

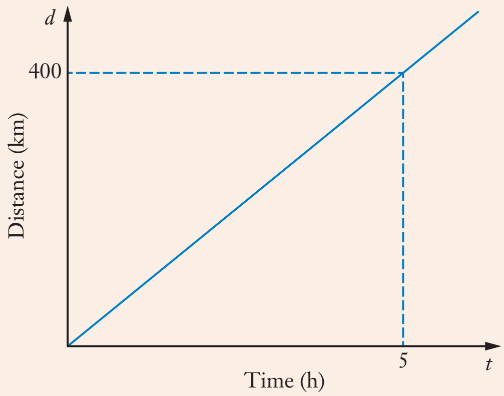
positive negative

What is the gradient of a horizontal line?

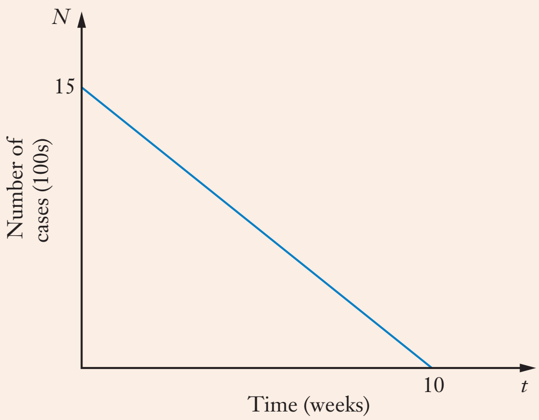
Can you find the gradient of a vertical line? Why?

EXAMPLE 1

a The graph shows the distance travelled by a car over time. Find the gradient and describe it as a rate.



b The graph shows the number of cases of flu reported in a town over several weeks. Find the gradient and describe it as a rate.



Solution

a
$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{400}{5}$$
$$= 80$$

The line is increasing, so it has a positive gradient.

This means that the car is travelling at a constant rate (speed) of 80 km/h.

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b $m = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{-1500}{10}$

$= -150$

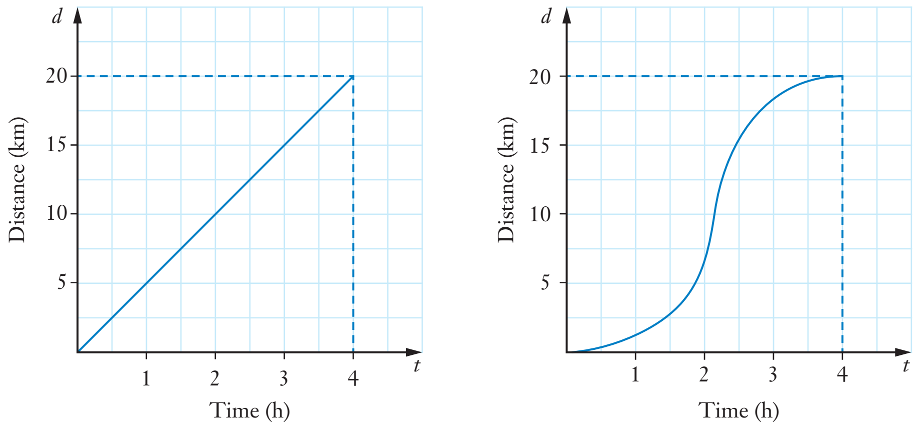
The line is decreasing, so it will have a negative gradient.

The 'rise' is a drop so it's negative.

This means that the rate is –150 cases/week, or the number of cases reported is decreasing by 150 cases/week.

CLASS DISCUSSION

The 2 graphs below show the distance that a bicycle travels over time. One is a straight line and the other is a curve.



Is the average speed of the bicycle the same in both cases? What is different about the speed in the 2 graphs?

How could you measure the speed in the second graph at any one time? Does it change? If so, how does it change?

We can start finding rates of change along a curve by looking at its shape and how it behaves. We started looking at this in Chapter 4, *Functions*.

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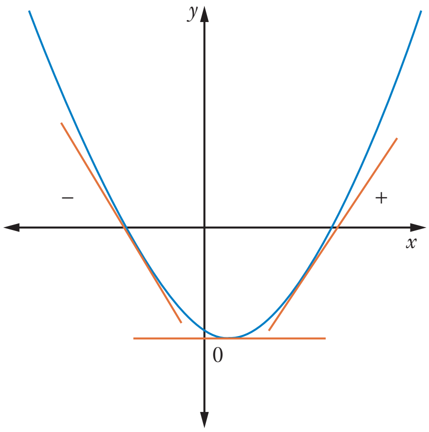
The gradient of a curve shows the **rate of change of *y*** as x changes. A **tangent** to a curve is a straight line that just touches the curve at one point. We can see where the gradient of a curve is positive, negative or zero by drawing **tangents to the curve** at different places around the curve and finding the gradients of the tangents.

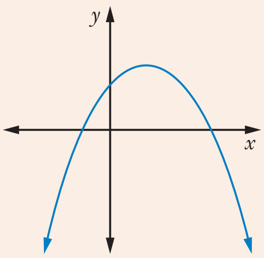
Notice that when the curve increases it has a positive gradient, when it decreases it has a negative gradient, and when it is a **turning point** the gradient is zero.



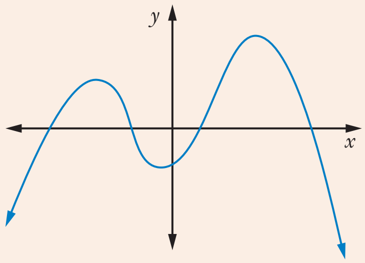
EXAMPLE 2

Copy each curve and write the sign of its gradient along the curve.

a



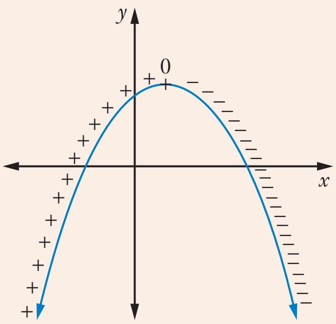
b



Solution

Where the curve increases, the gradient is positive. Where it decreases, it is negative. Where it is a turning point, it has a zero gradient.

a



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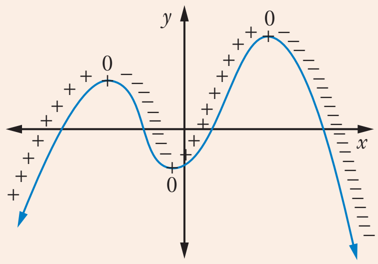

We find the gradient of a curve by measuring the **gradient of a tangent** to the curve at different points around the curve.

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We can then sketch the graph of these gradient values, which we call $y = f'(x)$, the **gradient function** or the **derivative function**.

EXAMPLE 3

a Make an accurate sketch of $f(x) = x^2$ on graph paper, or use graphing software.

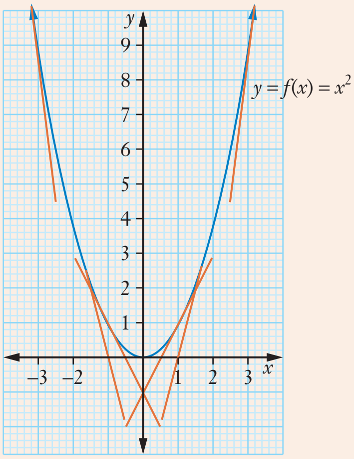
b Draw tangents to this curve at the points where $x = -3$, $x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

c Find the gradient of each of these tangents.

d Draw the graph of $y = f'(x)$ (the derivative or gradient function).

Solution

a and **b**



c At $x = -3$, $m = -6$

At $x = -2$, $m = -4$

At $x = -1$, $m = -2$

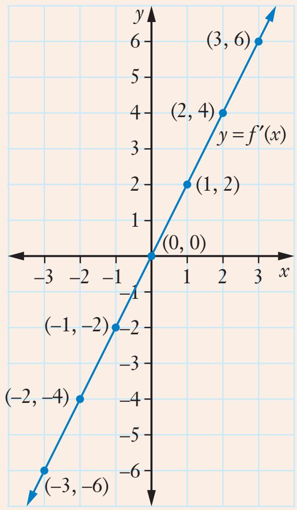
At $x = 0$, $m = 0$

At $x = 1$, $m = 2$

At $x = 2$, $m = 4$

At $x = 3$, $m = 6$

d Using the values from part **c**, $y = f'(x)$ is a linear function.



Notice in Example 3 that where $m > 0$, the gradient function is above the x-axis; where $m = 0$, the gradient function is on the x-axis; and where $m < 0$, the gradient function is below the x-axis. Since $m = f'(x)$, we can write the following:

Sketching gradient (derivative) functions

$f'(x) > 0$: gradient function is above the x-axis

$f'(x) < 0$: gradient function is below the x-axis

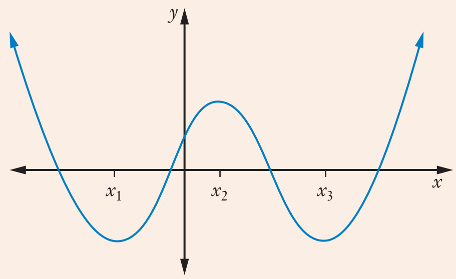
$f'(x) = 0$: gradient function is on the x-axis

Sketching gradient functions

EXAMPLE 4

Sketch a gradient function for each curve.

a



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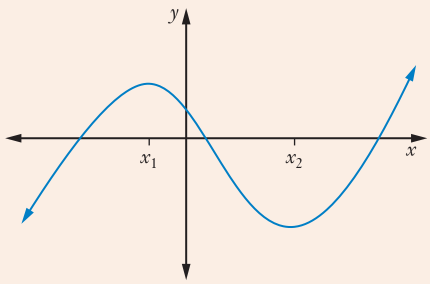
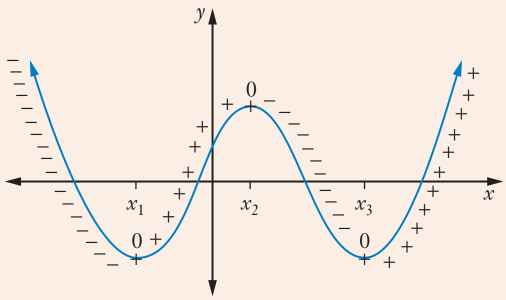
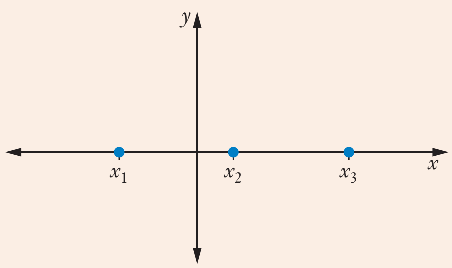


Solution

a First we mark in where the gradient is positive, negative and zero.



$f'(x) = 0$ at x_1, x_2 and x_3, so on the gradient graph these points will be on the x-axis (the x-intercepts of the gradient graph).



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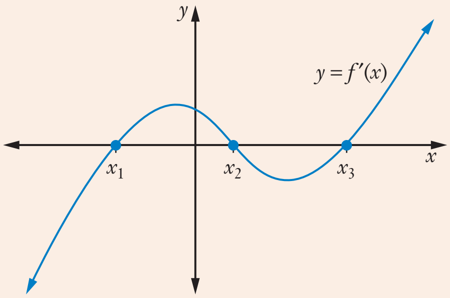
$f'(x) < 0$ to the left of x_1, so this part of the gradient graph will be below the x-axis.

$f'(x) > 0$ between x_1 and x_2, so the graph will be above the x-axis here.

$f'(x) < 0$ between x_2 and x_3, so the graph will be below the x-axis here.



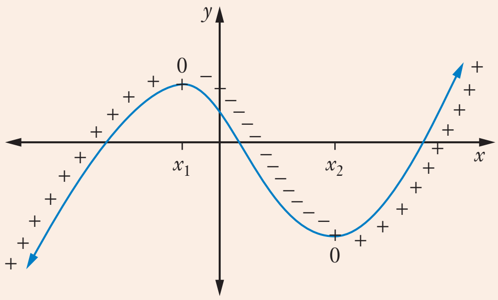
$f'(x) > 0$ to the right of x_3, so this part of the graph will be above the x-axis.

Sketching this information gives the graph of the gradient function $y = f'(x)$. Note that this is only a rough graph that shows the shape and sign rather than precise values.

b First mark in where the gradient is positive, negative and zero.

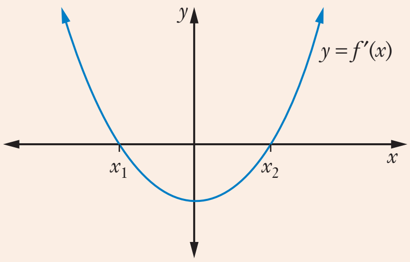
$f'(x) = 0$ at x_1 and x_2. These points will be the x-intercepts of the gradient function graph.

$f'(x) > 0$ to the left of x_1, so the graph will be above the x-axis here.



$f'(x) < 0$ between x_1 and x_2, so the graph will be below the x-axis here.

$f'(x) > 0$ to the right of x_2, so the graph will be above the x-axis here.



TECHNOLOGY

TANGENTS TO A CURVE

There are some excellent graphing software, online apps and websites that will draw tangents to a curve and sketch the gradient function.

Explore how to sketch gradient functions from the previous examples.

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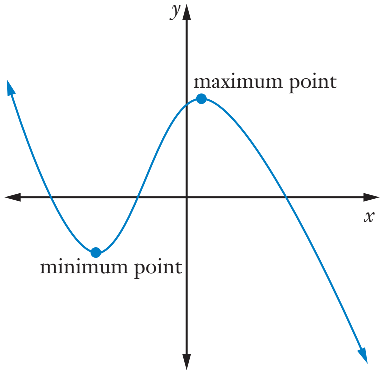
Stationary points

The points on a curve where the gradient $f'(x) = 0$ are called **stationary points** because the gradient there is neither increasing nor decreasing.

For example, the curve shown decreases to a **minimum turning point**, which is a type of **stationary point**. It then increases to a **maximum turning point** (also a stationary point) and then decreases again.



Exercise 8.01 Gradient of a curve

Sketch a gradient function for each curve.

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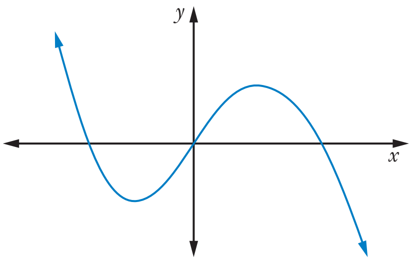

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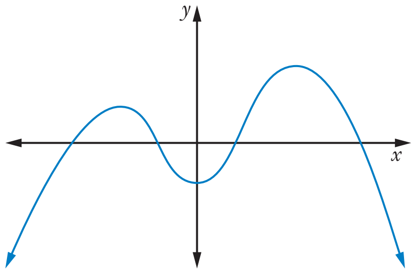

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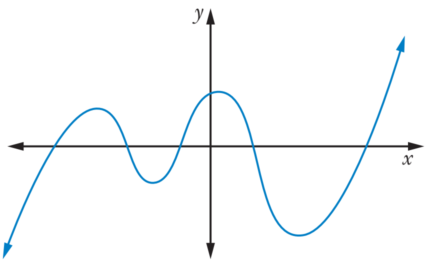

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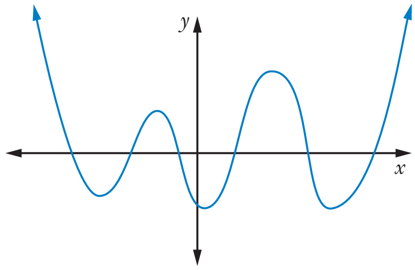

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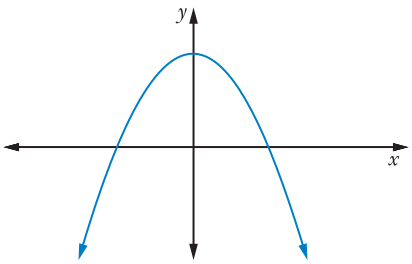

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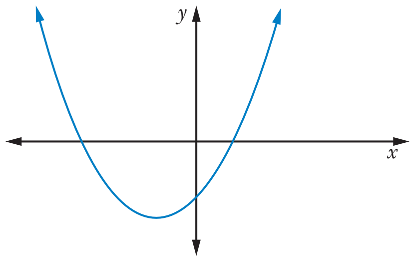

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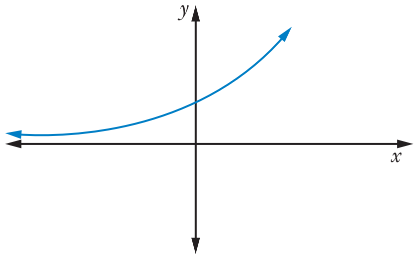

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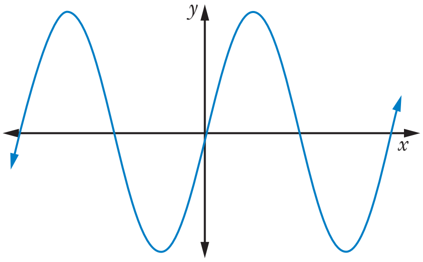

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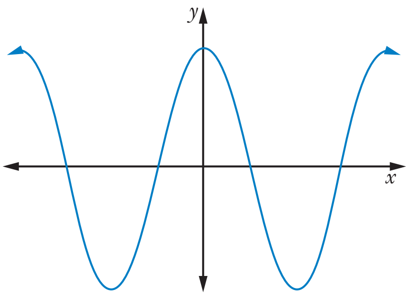

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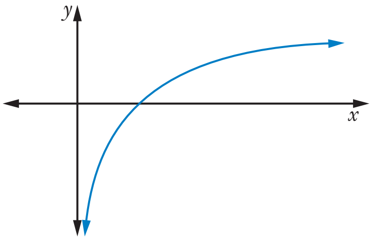


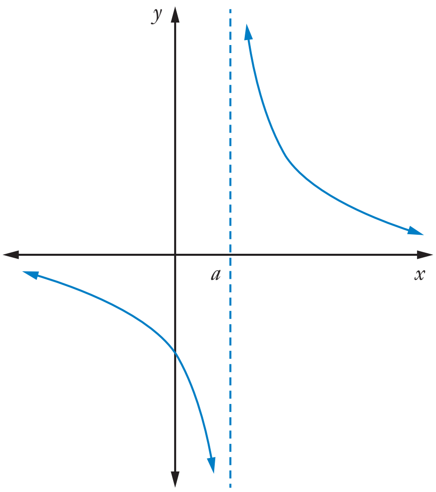
8.02 Differentiability

The process of finding the gradient function $y = f'(x)$ is called **differentiation**.

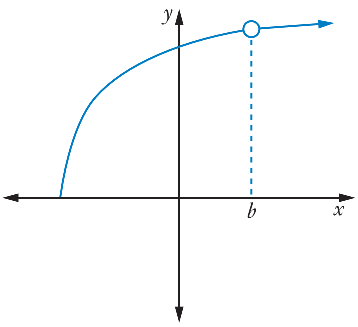
$y = f'(x)$ is called the **derivative function**, or just the **derivative**.

A function is **differentiable** at any point where it is continuous because we can find its gradient at that point. Linear, quadratic, cubic and other polynomial functions are differentiable at all points because their graphs are smooth and unbroken. A function is **not differentiable** at any point where it is **discontinuous**, where there is a gap or break in its graph.

This hyperbola is not differentiable at $x = a$ because the curve is discontinuous at this point.



This function is not differentiable at $x = b$ because the curve is discontinuous at this point.



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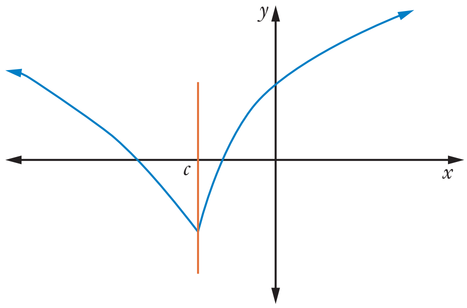
A function is also **not differentiable** where it is not smooth.

This function is not **differentiable** at $x = c$ since it is not smooth at that point. We cannot draw a unique tangent there so we cannot find the gradient of the function at that point.

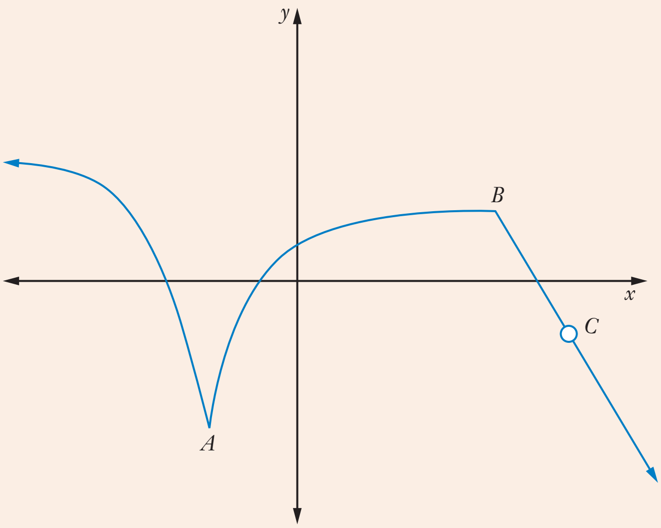


Differentiability at a point

A function $y = f(x)$ is **differentiable** at the point $x = a$ if its graph is **continuous** and **smooth** at $x = a$.

EXAMPLE 5

a Find all points where the function below is not differentiable.



b Is the function $f(x) = \begin{cases} x^2 & \text{for } x \geq 1 \\ 3x - 2 & \text{for } x < 1 \end{cases}$ differentiable at all points?

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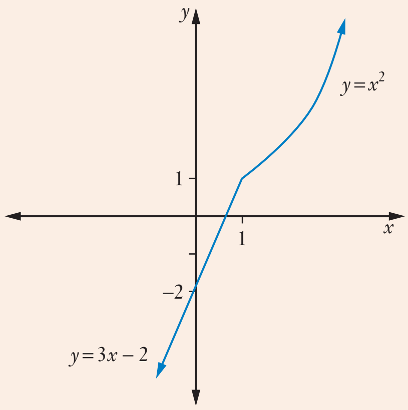
Solution

a The function is not differentiable at points A and B because the curve is not smooth at these points.

It is not differentiable at point C because the function is discontinuous at this point.

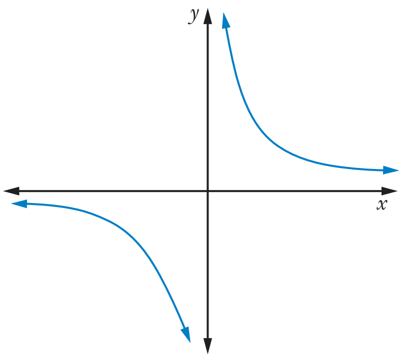
b Sketching this piecewise function shows that it is not smooth where the 2 parts meet, so it is not differentiable at $x = 1$.



Exercise 8.02 Differentiability

For each graph of a function, state any x values where the function is not differentiable.

1



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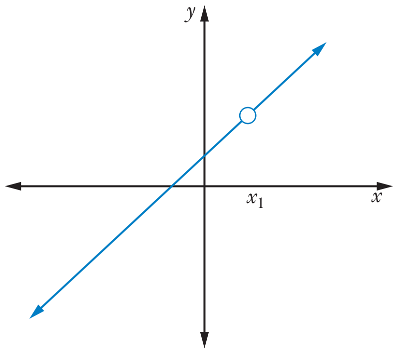

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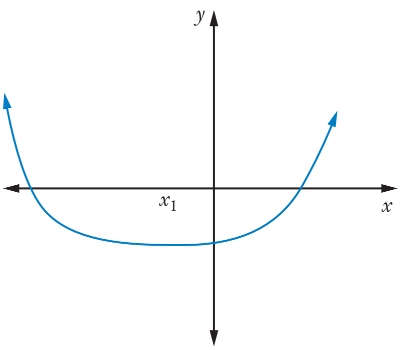

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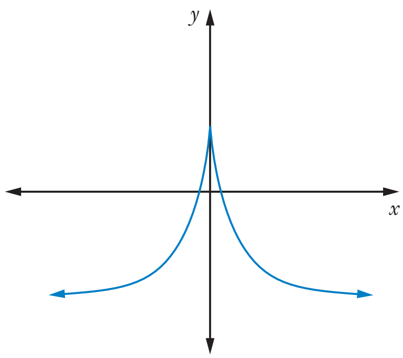

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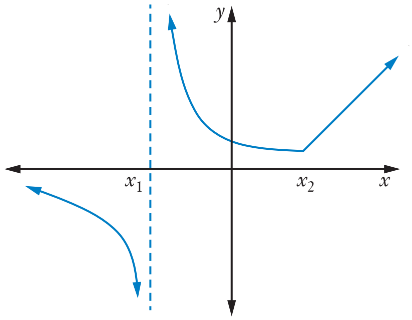


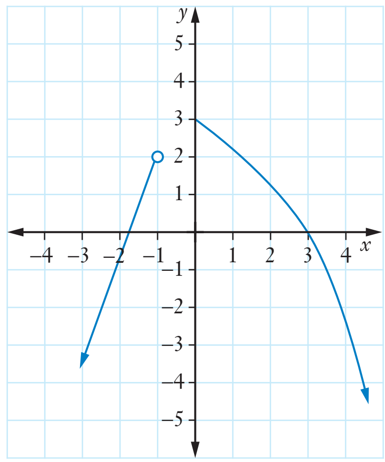
6 $f(x)=\dfrac{4}{x}$

7 $y=-\dfrac{1}{x+3}$

8 $f(x)=\begin{cases} x^3 & \text{for } x>2 \\ x+1 & \text{for } x\le 2 \end{cases}$

9 $f(x)=\begin{cases} 2x & \text{for } x>3 \\ 3 & \text{for } -2\le x\le 3 \\ 1-x^2 & \text{for } x<-2 \end{cases}$

10



11 $f(x)=\dfrac{|x|}{x}$

WS Differentiation from first principles

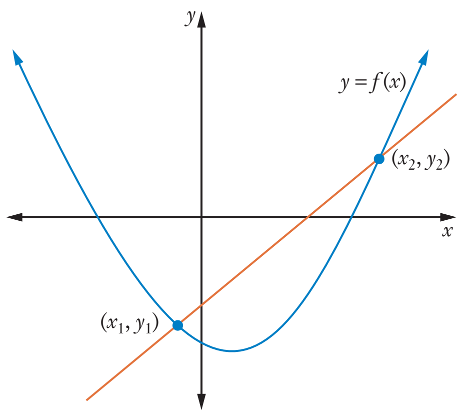
WS Limits

WS Finding derivatives from first principles

WS Rates of change - Gradients of secants

8.03 Differentiation from first principles

Gradient of a secant



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The line passing through the 2 points (x_1, y_1) and (x_2, y_2) on the graph of a function $y = f(x)$ is called a **secant**.

Differentiation from first principles

WS Limits

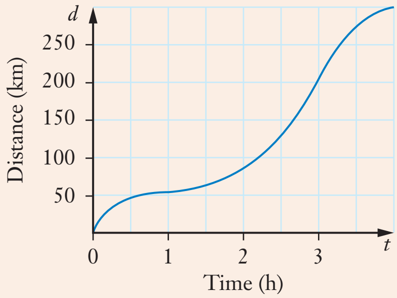
Gradient of the secant

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **gradient of a secant** gives the **average rate of change** between the 2 points.

EXAMPLE 6

a This graph shows the distance d in km that a car travels over time t in hours. After 1 hour the car has travelled 55 km and after 3 hours the car has travelled 205 km. Find the average speed of the car.



b Given the function $f(x) = x^2$, find the average rate of change between $x = 1$ and $x = 1.1$.

Solution

a Speed is the change in distance over time.

The gradient of the secant will give the average speed.

Average rate of change:

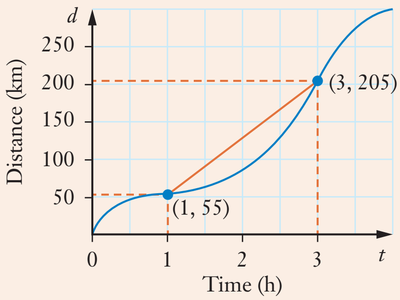
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{205 - 55}{3 - 1}$$

$$= \frac{150}{2}$$

$$= 75$$

So the average speed is 75 km/h.



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b When $x = 1$:

$f(1) = 1^2$

$= 1$

When $x = 1.1$:

$f(1.1) = 1.1^2$

$= 1.21$

So points are (1, 1) and (1.1, 1.21).

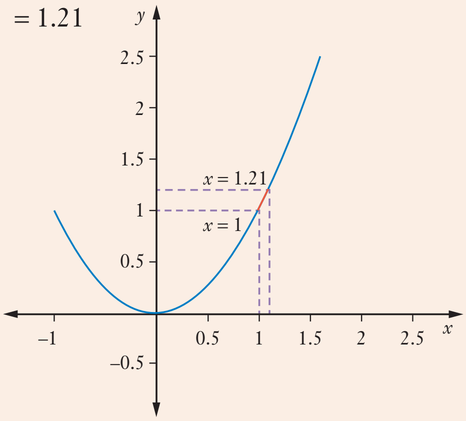
Average rate of change:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1.21 - 1}{1.1 - 1}$$

$$= 2.1$$

So the average rate of change is 2.1.



Notice that the secant (orange interval) is very close to the shape of the curve itself. This is because the 2 points chosen are close together.

Estimating the gradient of a tangent

By taking 2 points close together, the **average rate of change** is quite close to the gradient of the tangent to the curve at one of those points, which is called the **instantaneous rate of change** at that point.

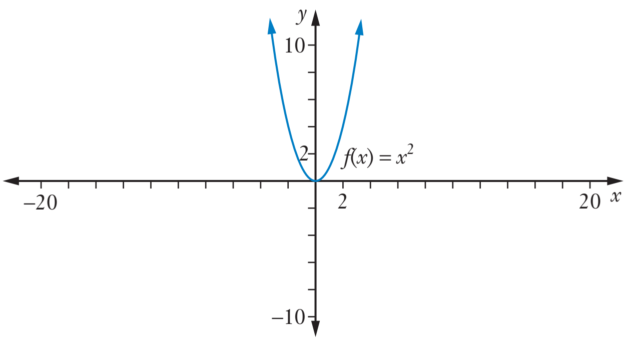
If you look at a close-up of a graph, you can get some idea of this concept. When the curve is magnified, any 2 points close together appear to be joined by a straight line. We say the curve is **locally straight**.

TECHNOLOGY

LOCALLY STRAIGHT CURVES

Use a graphics calculator or graphing software to sketch a curve and then zoom in on a section of the curve to see that it is locally straight.

For example, here is the parabola $y = x^2$.



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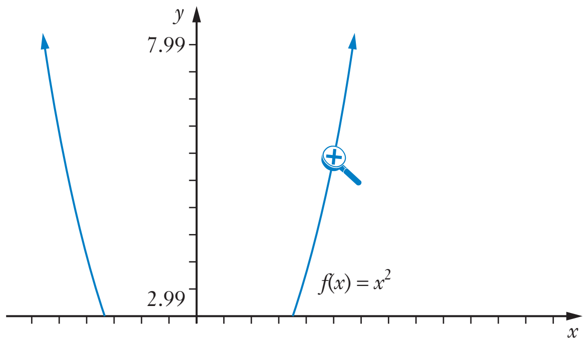
Notice how it looks straight when we zoom in on a point on the parabola.

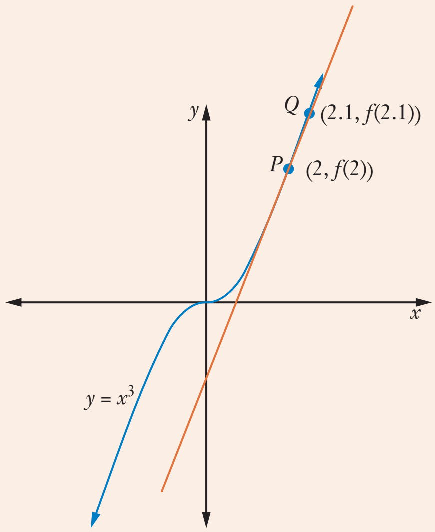


Use technology to sketch other curves and zoom in to show that they are locally straight.

We can calculate an approximate value for the gradient of the tangent at a point on a curve by taking another point close by, then calculating the gradient of the secant joining those 2 points.

EXAMPLE 7

a For $f(x) = x^3$, find the gradient of the secant PQ where P is the point on the curve where $x = 2$ and Q is another point on the curve where $x = 2.1$. Then choose different values for Q and use these results to estimate $f'(2)$, the gradient of the tangent to the curve at P.



b For the curve $y = x^2$, find the gradient of the secant AB where A is the point on the curve where $x = 5$ and point B is close to A. Find an estimate of the gradient of the tangent to the curve at A by using 3 different values for B.

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Solution

a P is $(2, f(2))$. Take different values of x for point Q, starting with $x = 2.1$, and find the gradient of the secant using $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Point Q	Gradient of secant PQ	Point Q	Gradient of secant PQ
$(2.1, f(2.1))$	$m = \frac{f(2.1) - f(2)}{2.1 - 2}$ $= \frac{2.1^3 - 2^3}{0.1}$ $= 12.61$	$(1.9, f(1.9))$	$m = \frac{f(1.9) - f(2)}{1.9 - 2}$ $= \frac{1.9^3 - 2^3}{-0.1}$ $= 11.41$
$(2.01, f(2.01))$	$m = \frac{f(2.01) - f(2)}{2.01 - 2}$ $= \frac{2.01^3 - 2^3}{0.01}$ $= 12.0601$	$(1.99, f(1.99))$	$m = \frac{f(1.99) - f(2)}{1.99 - 2}$ $= \frac{1.99^3 - 2^3}{-0.01}$ $= 11.9401$
$(2.001, f(2.001))$	$m = \frac{f(2.001) - f(2)}{2.001 - 2}$ $= \frac{2.001^3 - 2^3}{0.001}$ $= 12.006\,001$	$(1.999, f(1.999))$	$m = \frac{f(1.999) - f(2)}{1.999 - 2}$ $= \frac{1.999^3 - 2^3}{-0.001}$ $= 11.994\,001$

From these results, we can see that a good estimate for $f'(2)$, the gradient at P, is 12.

As $x \to 2, f'(2) \to 12$.

We use a special notation for **limits** to show this.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$
$$= 12$$

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b $A = (5, f(5))$

Take 3 different values of x for point B; for example, $x = 4.9$, $x = 5.1$ and $x = 5.01$.

$B = (4.9, f(4.9))$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(4.9) - f(5)}{4.9 - 5} \\ &= \frac{4.9^2 - 5^2}{-0.1} \\ &= 9.9 \end{aligned}$$

$B = (5.1, f(5.1))$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(5.1) - f(5)}{5.1 - 5} \\ &= \frac{5.1^2 - 5^2}{0.1} \\ &= 10.1 \end{aligned}$$

$B = (5.01, f(5.01))$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(5.01) - f(5)}{5.01 - 5} \\ &= \frac{5.01^2 - 5^2}{0.01} \\ &= 10.01 \end{aligned}$$

As $x \to 5, f'(5) \to 10$.

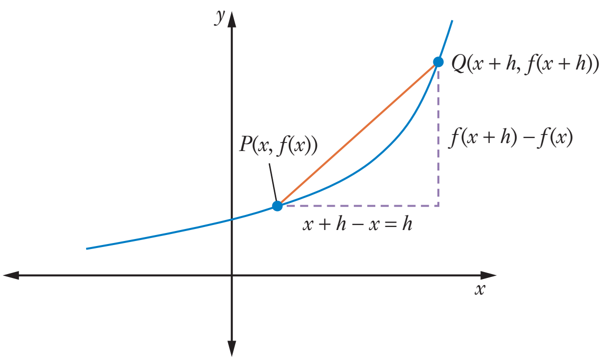
$$\begin{aligned} f'(5) &= \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} \\ &= 10 \end{aligned}$$

The difference quotient

We measure the instantaneous rate of change of any point on the graph of a function by using limits to find the gradient of the tangent to the curve at that point. This is called **differentiation from first principles**. Using the method from the examples above, we can find a general formula for the derivative function $y = f'(x)$.

We want to find the instantaneous rate of change or gradient of the tangent to a curve $y = f(x)$ at point $P(x, f(x))$.

We choose a second point Q close to P with coordinates $(x + h, f(x + h))$ where h is small.



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Now find the gradient of the secant PQ.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{x+h-x}$$

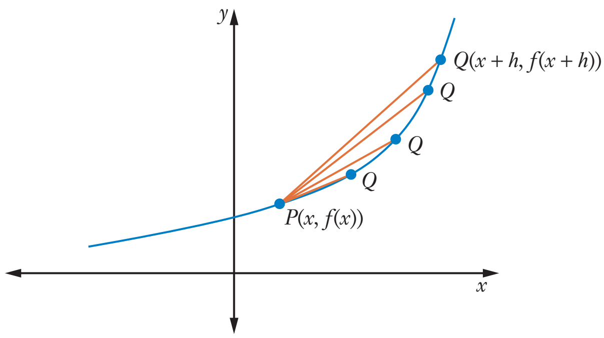
$$= \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h) - f(x)}{h}$ is called the **difference quotient** and it gives an **average rate of change**.

To find the gradient of the tangent at P, we make h smaller as shown, so that Q becomes closer and closer to P.

As h approaches 0, the gradient of the tangent becomes $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

We call this $f'(x)$ or $\frac{dy}{dx}$ or y'.



Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

INVESTIGATION

CALCULUS NOTATION

On p.412, we learned about the mathematicians Isaac Newton and Gottfried Leibniz. Newton used the notation $f'(x)$ for the derivative function while Leibniz used the notation $\frac{dy}{dx}$ where d stood for 'difference'. Can you see why he would have used this?

Use the Internet to explore the different notations used in calculus and where they came from.

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EXAMPLE 8

a Differentiate from first principles to find the gradient of the tangent to the curve $y = x^2 + 3$ at the point where $x = 1$.

b Differentiate $f(x) = 2x^2 + 7x - 3$ from first principles.

Solution

a

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 + 3$$

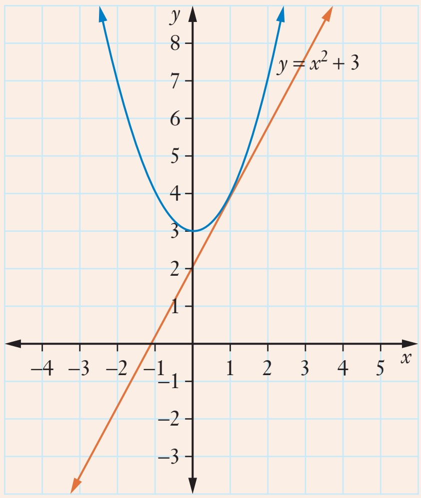
$$\begin{aligned} f(x+h) &= (x+h)^2 + 3 \\ &= x^2 + 2xh + h^2 + 3 \end{aligned}$$

Substitute $x = 1$:

$$\begin{aligned} f(1) &= 1^2 + 3 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(1+h) &= 1^2 + 2(1)h + h^2 + 3 \\ &= 4 + 2h + h^2 \end{aligned}$$

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{4 + 2h + h^2 - 4}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0} \frac{h(2+h)}{h} \\ &= \lim_{h \to 0} (2+h) \\ &= 2 + 0 \\ &= 2 \end{aligned}$$



So the gradient of the tangent to the curve $y = x^2 + 3$ at the point $(1, 4)$ is 2.

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b

$$\begin{aligned}
f(x) &= 2x^2 + 7x - 3 \\
f(x+h) &= 2(x+h)^2 + 7(x+h) - 3 \\
&= 2(x^2 + 2xh + h^2) + 7x + 7h - 3 \\
&= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 \\
f(x+h) - f(x) &= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - (2x^2 + 7x - 3) \\
&= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - 2x^2 - 7x + 3 \\
&= 4xh + 2h^2 + 7h \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 + 7h}{h} \\
&= \lim_{h\to 0} \frac{h(4x + 2h + 7)}{h} \\
&= \lim_{h\to 0} (4x + 2h + 7) \\
&= 4x + 0 + 7 \\
&= 4x + 7
\end{aligned}$$

So the gradient function (derivative) of $f(x) = 2x^2 + 7x - 3$ is $f'(x) = 4x + 7$.

Exercise 8.03 Differentiation from first principles

1 **a** For the curve $y = x^4 + 1$, find the gradient of the secant between the point (1, 2) and the point where $x = 1.01$.

b Find the gradient of the secant between (1, 2) and the point where $x = 0.999$ on the curve.

c Use these results to find an approximation to the gradient of the tangent to the curve $y = x^4 + 1$ at the point (1, 2).

2 For the function $f(x) = x^3 + x$, find the average rate of change between the point (2, 10) and the point on the curve where:

a $x = 2.1$ **b** $x = 2.01$ **c** $x = 1.99$

d Hence find an approximation to the gradient of the tangent at the point (2, 10).

3 For the function $f(x) = x^2 - 4$, find the gradient of the tangent at point P where $x = 3$ by selecting points near P and finding the gradient of the secant.

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4 A function is given by $f(x) = x^2 + x + 5$.

a Find $f(2)$.

b Find $f(2 + h)$.

c Find $f(2 + h) - f(2)$.

d Show that $\dfrac{f(2+h)-f(2)}{h} = 5 + h$.

e Find $f'(2)$.

5 Given the curve $f(x) = 4x^2 - 3$, find:

a $f(-1)$

b $f(-1 + h) - f(-1)$

c the gradient of the tangent to the curve at the point where $x = -1$.

6 For the parabola $y = x^2 - 1$, find:

a $f(3)$

b $f(3 + h) - f(3)$

c $f'(3)$.

7 For the function $f(x) = 4 - 3x - 5x^2$, find:

a $f'(1)$

b the gradient of the tangent at the point $(-2, -10)$.

8 If $f(x) = x^2$:

a find $f(x + h)$

b show that $f(x + h) - f(x) = 2xh + h^2$

c show that $\dfrac{f(x+h)-f(x)}{h} = 2x + h$

d show that $f'(x) = 2x$.

9 A function is given by $f(x) = 2x^2 - 7x + 3$.

a Show that $f(x + h) = 2x^2 + 4xh + 2h^2 - 7x - 7h + 3$.

b Show that $f(x + h) - f(x) = 4xh + 2h^2 - 7h$.

c Show that $\dfrac{f(x+h)-f(x)}{h} = 4x + 2h - 7$.

d Find $f'(x)$.

10 Differentiate from first principles to find the gradient of the tangent to the curve:

a $f(x) = x^2$ at the point where $x = 1$

b $y = x^2 + x$ at the point $(2, 6)$

c $f(x) = 2x^2 - 5$ at the point where $x = -3$

d $y = 3x^2 + 3x + 1$ at the point where $x = 2$

e $f(x) = x^2 - 7x - 4$ at the point $(-1, 4)$.

11 Find the derivative function for each function from first principles.

a $f(x) = x^2$

b $y = x^2 + 5x$

c $f(x) = 4x^2 - 4x - 3$

d $y = 5x^2 - x - 1$

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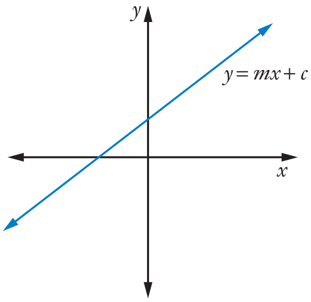
WS
Derivatives of linear products
WS
Derivatives of polynomials

8.04 Short methods of differentiation

Derivative of x^n

Remember that the gradient of a straight line $y = mx + c$ is m. The tangent to the line is the line itself, so the gradient of the tangent is m everywhere along the line.

So if $y = mx$, $\frac{dy}{dx} = m$.

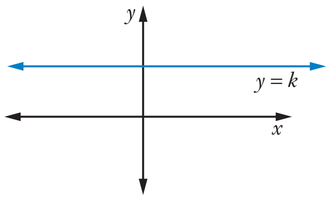


Derivative of *kx*

$$\frac{d}{dx}(kx) = k$$

A horizontal line $y = k$ has a gradient of zero.

So if $y = k$, $\frac{dy}{dx} = 0$.



Derivative of *k*

$$\frac{d}{dx}(k) = 0$$

TECHNOLOGY

DIFFERENTIATION OF POWERS OF *X*

Find an approximation to the derivative of power functions such as $y = x^2, y = x^3, y = x^4$, $y = x^5$ by drawing the graph of $y = \frac{f(x + 0.01) - f(x)}{0.01}$. You could use a graphics calculator or graphing software/website to sketch the derivative for these functions and find its equation. Can you find a pattern? Could you predict what the result would be for x^n?

When differentiating $y = x^n$ from first principles, a simple pattern appears:

- For $y = x$, $f'(x) = 1x^0 = 1$
- For $y = x^2$, $f'(x) = 2x^1 = 2x$
- For $y = x^3$, $f'(x) = 3x^2$
- For $y = x^4$, $f'(x) = 4x^3$
- For $y = x^5$, $f'(x) = 5x^4$

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Derivative of x^n

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.

There are some more properties of differentiation.

Derivative of kx^n

$$\frac{d}{dx}(kx^n) = knx^{n-1}$$

More generally:

Derivative of a constant multiple of a function

$$\frac{d}{dx}\left(kf(x)\right) = kf'(x)$$

EXAMPLE 9

a Find the derivative of $3x^8$.

b Differentiate $f(x) = 7x^3$.

Solution

a

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}(3x^8) = 3 \times 8x^{8-1}$$

$$= 24x^7$$

b

$$f'(x) = knx^{n-1}$$

$$f'(x) = 7 \times 3x^{3-1}$$

$$= 21x^2$$

If there are several terms in an expression, we differentiate each one separately.

Derivative of a sum of functions

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

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EXAMPLE 10

a Differentiate $x^3 + x^4$.

b Find the derivative of $7x$.

c Differentiate $f(x) = x^4 - x^3 + 5$.

d Find the derivative of $y = 4x^7$.

e If $f(x) = 2x^5 - 7x^3 + 5x - 4$, evaluate $f'(-1)$.

f Find the derivative of $f(x) = 2x^2(3x - 7)$.

g Find the derivative of $\dfrac{3x^2 + 5x}{2x}$.

h Differentiate $S = 6r^2 - 12r$ with respect to r.

Solution

a $\dfrac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3$

b $\dfrac{d}{dx}(7x) = 7$

c
$$\begin{aligned} f'(x) &= 4x^3 - 3x^2 + 0 \\ &= 4x^3 - 3x^2 \end{aligned}$$

d
$$\begin{aligned} \frac{dy}{dx} &= 4 \times 7x^6 \\ &= 28x^6 \end{aligned}$$

e
$$\begin{aligned} f'(x) &= 10x^4 - 21x^2 + 5 \\ f'(-1) &= 10(-1)^4 - 21(-1)^2 + 5 \\ &= -6 \end{aligned}$$

f Expand first.
$$\begin{aligned} f(x) &= 2x^2(3x - 7) \\ &= 6x^3 - 14x^2 \\ f'(x) &= 18x^2 - 28x \end{aligned}$$

g Simplify first.
$$\begin{aligned} \frac{3x^2 + 5x}{2x} &= \frac{3x^2}{2x} + \frac{5x}{2x} \\ &= \frac{3x}{2} + \frac{5}{2} \end{aligned}$$
$$\begin{aligned} \frac{d}{dx}\left(\frac{3x^2 + 5x}{2x}\right) &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

h Differentiating with respect to r rather than x:
$$\begin{aligned} S &= 6r^2 - 12r \\ \frac{dS}{dr} &= 12r - 12 \end{aligned}$$

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INVESTIGATION

FAMILIES OF CURVES

1 Differentiate:

a $x^2 + 1$ **b** $x^2 - 3$ **c** $x^2 + 7$

d x^2 **e** $x^2 + 20$ **f** $x^2 - 100$

What do you notice?

2 Differentiate:

a $x^3 + 5$ **b** $x^3 + 11$ **c** $x^3 - 1$

d $x^3 - 6$ **e** x^3 **f** $x^3 + 15$

What do you notice?

These groups of functions are families because they have the same derivatives. Can you find others?

Exercise 8.04 Short methods of differentiation

1 Differentiate:

a $x + 2$ **b** $5x - 9$ **c** $x^2 + 3x + 4$

d $5x^2 - x - 8$ **e** $x^3 + 2x^2 - 7x - 3$ **f** $2x^3 - 7x^2 + 7x - 1$

g $3x^4 - 2x^2 + 5x$ **h** $x^6 - 5x^5 - 2x^4$ **i** $2x^5 - 4x^3 + x^2 - 2x + 4$

j $4x^{10} - 7x^9$

2 Find the derivative of:

a $x(2x + 1)$ **b** $(2x - 3)^2$ **c** $(x + 4)(x - 4)$

d $(2x^2 - 3)^2$ **e** $(2x + 5)(x^2 - x + 1)$

3 Find the derivative of:

a $\frac{x^2}{6} - x$ **b** $\frac{x^4}{2} - \frac{x^3}{3} + 4$ **c** $\frac{1}{3}x^6(x^2 - 3)$

d $\frac{2x^3 + 5x}{x}$ **e** $\frac{x^2 + 2x}{4x}$ **f** $\frac{2x^5 - 3x^4 + 6x^3 - 2x^2}{3x^2}$

4 Find $f'(x)$ when $f(x) = 8x^2 - 7x + 4$.

5 If $y = x^4 - 2x^3 + 5$, find $\frac{dy}{dx}$ when $x = -2$.

6 Find $\frac{dy}{dx}$ if $y = 6x^{10} - 5x^8 + 7x^5 - 3x + 8$.

7 If $s = 5t^2 - 20t$, find $\frac{ds}{dt}$.

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8 Find $g'(x)$ given $g(x) = 5x^4$.

9 Find $\frac{dv}{dt}$ when $v = 15t^2 - 9$.

10 If $h = 40t - 2t^2$, find $\frac{dh}{dt}$.

11 Given $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$.

12 If $f(x) = 2x^3 - 3x + 4$, evaluate $f'(1)$.

13 Given $f(x) = x^2 - x + 5$, evaluate:

a $f'(3)$ **b** $f'(-2)$ **c** x when $f'(x) = 7$

14 If $y = x^3 - 7$, evaluate:

a the derivative when $x = 2$ **b** x when $\frac{dy}{dx} = 12$

15 Evaluate $g'(2)$ when $g(t) = 3t^3 - 4t^2 - 2t + 1$.

DID YOU KNOW?

Motion and calculus

Galileo (1564–1642) was very interested in the behaviour of bodies in motion. He dropped stones from the Leaning Tower of Pisa to try to prove that they would fall with equal speed. He rolled balls down slopes to prove that they move with uniform speed until friction slows them down. He showed that a body moving through the air follows a curved path at a fairly constant speed.

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Galileo

John Wallis (1616–1703) continued this study with his publication *Mechanica, sive Tractatus de Motu Geometricus*. He applied mathematical principles to the laws of motion and stimulated interest in the subject of mechanics.

Soon after Wallis' publication, **Christiaan Huygens** (1629–1695) wrote *Horologium Oscillatorium sive de Motu Pendulorum*, in which he described various mechanical principles. He invented the pendulum clock, improved the telescope and investigated circular motion and the descent of heavy bodies.

These three mathematicians provided the foundations of mechanics. **Sir Isaac Newton** (1642–1727) used calculus to increase the understanding of the laws of motion. He also used these concepts as a basis for his theories on gravity and inertia.

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8.05 Derivatives and indices

INVESTIGATION

DERIVATIVES AND INDICES

1 **a** Show that $\frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$.

b Hence differentiate $y = \frac{1}{x}$ from first principles.

c Differentiate $y = x^{-1}$ using the formula. Do you get the same answer as in part **b**?

2 **a** Show that $\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right) = h$.

b Hence differentiate $y = \sqrt{x}$ from first principles.

c Differentiate $y = x^{\frac{1}{2}}$ and show that this gives the same answer as in part **b**.

EXAMPLE 11

a Differentiate $f(x) = 7\sqrt[3]{x}$.

b Find the derivative of $y = \frac{4}{x^2}$ at the point where $x = 2$.

Solution

a $f(x) = 7\sqrt[3]{x} = 7x^{\frac{1}{3}}$

Convert the function to a power of *x* first.

$$f'(x) = 7 \times \frac{1}{3} x^{\frac{1}{3} - 1}$$

$$= \frac{7}{3} x^{-\frac{2}{3}}$$

$$= \frac{7}{3} \times \frac{1}{x^{\frac{2}{3}}}$$

$$= \frac{7}{3} \times \frac{1}{\sqrt[3]{x^2}}$$

$$= \frac{7}{3\sqrt[3]{x^2}}$$

b $y = \frac{4}{x^2}$

$$= 4x^{-2}$$

$$\frac{dy}{dx} = -8x^{-3}$$

$$= -\frac{8}{x^3}$$

When $x = 2$:

$$\frac{dy}{dx} = -\frac{8}{2^3}$$

$$= -1$$

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Exercise 8.05 Derivatives and indices

1 Differentiate:

a x^{-3} **b** $x^{1.4}$ **c** $6x^{0.2}$ **d** $x^{\frac{1}{2}}$

e $2x^{\frac{1}{2}} - 3x^{-1}$ **f** $3x^{\frac{1}{3}}$ **g** $8x^{\frac{3}{4}}$ **h** $-2x^{-\frac{1}{2}}$

2 Find the derivative function.

a $\frac{1}{x}$ **b** $5\sqrt{x}$ **c** $\sqrt[6]{x}$ **d** $\frac{2}{x^5}$

e $-\frac{5}{x^3}$ **f** $\frac{1}{\sqrt{x}}$ **g** $\frac{1}{2x^6}$ **h** $x\sqrt{x}$

i $\frac{2}{3x}$ **j** $\frac{1}{4x^2} + \frac{3}{x^4}$

3 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 27$.

4 If $x = \frac{12}{t}$, find $\frac{dx}{dt}$ when $t = 2$.

5 A function is given by $f(x) = \sqrt[4]{x}$. Evaluate $f'(16)$.

6 Find the derivative of $y = \frac{3}{2x^2}$ at the point $\left(1, 1\frac{1}{2}\right)$.

7 Find $\frac{dy}{dx}$ if $y = \left(x + \sqrt{x}\right)^2$.

8 A function $f(x) = \frac{\sqrt{x}}{2}$ has a tangent at (4, 1). Find its gradient.

9 **a** Differentiate $\frac{\sqrt{x}}{x}$.

b Hence find the derivative of $y = \frac{\sqrt{x}}{x}$ at the point where $x = 4$.

10 The function $f(x) = 3\sqrt{x}$ has $f'(x) = \frac{3}{4}$ at $x = a$. Find a.

11 The hyperbola $y = \frac{2}{x}$ has 2 tangents with gradient $-\frac{2}{25}$. Find the points where these tangents touch the hyperbola.

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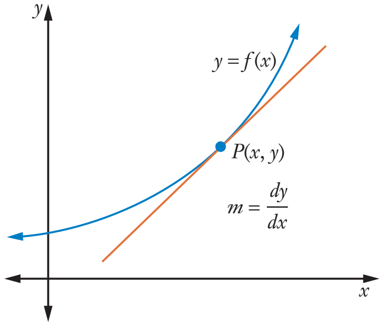
8.06 Tangents and normals

Tangents to a curve

Remember that the derivative is a function that gives the instantaneous rate of change or gradient of the tangent to the curve.

A tangent is a line so we can use the formula $y = mx + c$ or $y - y_1 = m(x - x_1)$ to find its equation.



Tangents and normals

Equation of a tangent

Slopes of curves

Tangents to a curve

EXAMPLE 12

a Find the gradient of the tangent to the parabola $y = x^2 + 1$ at the point (1, 2).

b Find values of x for which the gradient of the tangent to the curve $y = 2x^3 - 6x^2 + 1$ is equal to 18.

c Find the equation of the tangent to the curve $y = x^4 - 3x^3 + 7x - 2$ at the point (2, 4).

Solution

a The gradient of a tangent to a curve is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x + 0$$
$$= 2x$$

Substitute $x = 1$ from the point (1, 2):

$$\frac{dy}{dx} = 2(1)$$
$$= 2$$

So the gradient of the tangent at (1, 2) is 2.

b $\frac{dy}{dx} = 6x^2 - 12x$

Gradient is 18 so $\frac{dy}{dx} = 18$.

$$18 = 6x^2 - 12x$$
$$0 = 6x^2 - 12x - 18$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$\therefore x = 3, -1$$

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c $$\frac{dy}{dx} = 4x^3 - 9x^2 + 7$$

At $(2, 4)$, $\frac{dy}{dx} = 4(2)^3 - 9(2)^2 + 7$

$= 3$

So the gradient of the tangent at $(2, 4)$ is 3.

Equation of the tangent:

$y - y_1 = m(x - x_1)$

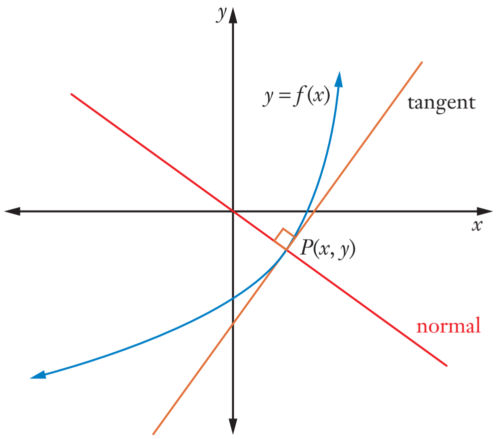
$y - 4 = 3(x - 2)$

$= 3x - 6$

$y = 3x - 2$ or $3x - y - 2 = 0$

Normals to a curve

The **normal** is a straight line **perpendicular** to the tangent at the same point of contact with the curve.



Remember the rule for perpendicular lines from Chapter 4, *Functions*:

Gradients of perpendicular lines

If 2 lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$ or $m_2 = -\frac{1}{m_1}$.

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EXAMPLE 13

a Find the gradient of the normal to the curve $y = 2x^2 - 3x + 5$ at the point where $x = 4$.

b Find the equation of the normal to the curve $y = x^3 + 3x^2 - 2x - 1$ at $(-1, 3)$.

Solution

a $\frac{dy}{dx} = 4x - 3$

When $x = 4$:

$$\frac{dy}{dx} = 4 \times 4 - 3$$
$$= 13$$

So $m_1 = 13$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$$13m_2 = -1$$
$$m_2 = -\frac{1}{13}$$

So the gradient of the normal is $-\frac{1}{13}$.

b $\frac{dy}{dx} = 3x^2 + 6x - 2$

When $x = -1$:

$$\frac{dy}{dx} = 3(-1)^2 + 6(-1) - 2$$
$$= -5$$

So $m_1 = -5$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$$-5m_2 = -1$$
$$m_2 = \frac{1}{5}$$

So the gradient of the normal is $\frac{1}{5}$.

Equation of the normal: $y - y_1 = m(x - x_1)$

$$y - 3 = \frac{1}{5}(x - (-1))$$
$$5y - 15 = x + 1$$
$$x - 5y + 16 = 0$$

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Exercise 8.06 Tangents and normals

1 Find the gradient of the tangent to the curve:

a $y = x^3 - 3x$ at the point where $x = 5$

b $f(x) = x^2 + x - 4$ at the point $(-7, 38)$

c $f(x) = 5x^3 - 4x - 1$ at the point where $x = -1$

d $y = 5x^2 + 2x + 3$ at $(-2, 19)$

e $y = 2x^9$ at the point where $x = 1$

f $f(x) = x^3 - 7$ at the point where $x = 3$

g $v = 2t^2 + 3t - 5$ at the point where $t = 2$

h $Q = 3r^3 - 2r^2 + 8r - 4$ at the point where $r = 4$

i $h = t^4 - 4t$ where $t = 0$

j $f(t) = 3t^5 - 8t^3 + 5t$ at the point where $t = 2$.

2 Find the gradient of the normal to the curve:

a $f(x) = 2x^3 + 2x - 1$ at the point where $x = -2$

b $y = 3x^2 + 5x - 2$ at $(-5, 48)$

c $f(x) = x^2 - 2x - 7$ at the point where $x = -9$

d $y = x^3 + x^2 + 3x - 2$ at $(-4, -62)$

e $f(x) = x^{10}$ at the point where $x = -1$

f $y = x^2 + 7x - 5$ at $(-7, -5)$

g $A = 2x^3 + 3x^2 - x + 1$ at the point where $x = 3$

h $f(a) = 3a^2 - 2a - 6$ at the point where $a = -3$.

i $V = h^3 - 4h + 9$ at $(2, 9)$

j $g(x) = x^4 - 2x^2 + 5x - 3$ at the point where $x = -1$.

3 Find the gradient of **i** the tangent and **ii** the normal to the curve:

a $y = x^2 + 1$ at $(3, 10)$

b $f(x) = 5 - x^2$ where $x = -4$

c $y = 2x^5 - 7x^2 + 4$ where $x = -1$

d $p(x) = x^6 - 3x^4 - 2x + 8$ where $x = 1$

e $f(x) = 4 - x - x^2$ at $(-6, 26)$

4 Find the equation of the tangent to the curve:

a $y = x^4 - 5x + 1$ at $(2, 7)$

b $f(x) = 5x^3 - 3x^2 - 2x + 6$ at $(1, 6)$

c $y = x^2 + 2x - 8$ at $(-3, -5)$

d $y = 3x^3 + 1$ where $x = 2$

e $v = 4t^4 - 7t^3 - 2$ where $t = 2$

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5 Find the equation of the normal to the curve:

a $f(x) = x^3 - 3x + 5$ at $(3, 23)$

b $y = x^2 - 4x - 5$ at $(-2, 7)$

c $f(x) = 7x - 2x^2$ where $x = 6$

d $y = 7x^2 - 3x - 3$ at $(-3, 69)$

e $y = x^4 - 2x^3 + 4x + 1$ where $x = 1$

6 Find the equation of **i** the tangent and **ii** the normal to the curve:

a $f(x) = 4x^2 - x + 8$ at $(1, 11)$

b $y = x^3 - 2x^2 - 5x$ at $(-3, -30)$

c $F(x) = x^5 - 5x^3$ where $x = 1$

d $y = x^2 - 8x + 7$ at $(3, -8)$.

7 For the curve $y = x^3 - 27x - 5$, find values of x for which $\frac{dy}{dx} = 0$.

8 Find the coordinates of the points at which the curve $y = x^3 + 1$ has a tangent with a gradient of 3.

9 A function $f(x) = x^2 + 4x - 12$ has a tangent with a gradient of -6 at point P on the curve. Find the coordinates of P.

10 The tangent at point P on the curve $y = 4x^2 + 1$ is parallel to the x-axis. Find the coordinates of P.

11 Find the coordinates of point Q where the tangent to the curve $y = 5x^2 - 3x$ is parallel to the line $7x - y + 3 = 0$.

12 Find the coordinates of point S where the tangent to the curve $y = x^2 + 4x - 1$ is perpendicular to the line $4x + 2y + 7 = 0$.

13 The curve $y = 3x^2 - 4$ has a gradient of 6 at point A.

a Find the coordinates of A.

b Find the equation of the tangent to the curve at A.

14 A function $h = 3t^2 - 2t + 5$ has a tangent at the point where $t = 2$. Find the equation of the tangent.

15 A function $f(x) = 2x^2 - 8x + 3$ has a tangent parallel to the line $4x - 2y + 1 = 0$ at point P. Find the equation of the tangent at P.

16 Find the equation of the tangent to the curve $y = \frac{1}{x^3}$ at $\left(2, \frac{1}{8}\right)$.

17 Find the equation of the tangent to $f(x) = 6\sqrt{x}$ at the point where $x = 9$.

18 Find the equation of the tangent to the curve $y = \frac{4}{x}$ at $\left(8, \frac{1}{2}\right)$.

19 If the gradient of the tangent to $y = \sqrt{x}$ is $\frac{1}{6}$ at point A, find the coordinates of A.

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8.07 Chain rule

WS Chain rule

We looked at composite functions in Chapter 7, *Further functions*.

The **chain rule** is a method for differentiating composite functions. It is also called the **composite function** rule or the '**function of a function**' **rule**.

The chain rule

If a function y can be written as a composite function where $y = f(u(x))$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule

EXAMPLE 14

Differentiate:

a $y = (5x + 4)^7$ **b** $y = (3x^2 + 2x - 1)^9$ **c** $y = \sqrt{3 - x}$

Solution

a Let $u = 5x + 4$

Then $\dfrac{du}{dx} = 5$

$y = u^7$

$\therefore \dfrac{dy}{du} = 7u^6$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 7u^6 \times 5\\ &= 35u^6\\ &= 35(5x + 4)^6\end{aligned}$$

b Let $u = 3x^2 + 2x - 1$

Then $\dfrac{du}{dx} = 6x + 2$

$y = u^9$

$\therefore \dfrac{dy}{du} = 9u^8$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 9u^8 \times (6x + 2)\\ &= 9(3x^2 + 2x - 1)^8(6x + 2)\\ &= 9(6x + 2)(3x^2 + 2x - 1)^8\end{aligned}$$

c $y = \sqrt{3 - x} = (3 - x)^{\frac{1}{2}}$

Let $u = 3 - x$

Then $\dfrac{du}{dx} = -1$

$y = u^{\frac{1}{2}}$

$\dfrac{dy}{du} = \dfrac{1}{2}u^{-\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{2}u^{-\frac{1}{2}} \times (-1)\\ &= -\frac{1}{2}(3 - x)^{-\frac{1}{2}}\\ &= -\frac{1}{2\sqrt{3 - x}}\end{aligned}$$

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You might see a pattern when using the chain rule. The derivative of a composite function is the product of the derivatives of 2 functions.

The derivative of $[f(x)]^n$

$$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$$

EXAMPLE 15

Differentiate:

a $y = (8x^3 - 1)^5$ **b** $y = (3x + 8)^{11}$ **c** $y = \frac{1}{(6x+1)^2}$

Solution

a
$$\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$$
$$= 24x^2 \times 5(8x^3 - 1)^4$$
$$= 120x^2(8x^3 - 1)^4$$

b
$$\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$$
$$= 3 \times 11(3x + 8)^{10}$$
$$= 33(3x + 8)^{10}$$

c
$$y = \frac{1}{(6x+1)^2} = (6x+1)^{-2}$$
$$\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$$
$$= 6 \times (-2)(6x + 1)^{-3}$$
$$= -12(6x + 1)^{-3}$$
$$= -\frac{12}{(6x+1)^3}$$

Exercise 8.07 Chain rule

1 Differentiate:

a $y = (x + 3)^4$ **b** $y = (2x - 1)^3$ **c** $y = (5x^2 - 4)^7$

d $y = (8x + 3)^6$ **e** $y = (1 - x)^5$ **f** $y = 3(5x + 9)^9$

g $y = 2(x - 4)^2$ **h** $y = (2x^3 + 3x)^4$ **i** $y = (x^2 + 5x - 1)^8$

j $y = (x^6 - 2x^2 + 3)^6$ **k** $y = (3x - 1)^{\frac{1}{2}}$ **l** $y = (4 - x)^{-2}$

m $y = (x^2 - 9)^{-3}$ **n** $y = (5x + 4)^{\frac{1}{3}}$ **o** $y = (x^3 - 7x^2 + x)^{\frac{3}{4}}$

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p $y = \sqrt{3x+4}$ **q** $y = \dfrac{1}{5x-2}$ **r** $y = \dfrac{1}{(x^2+1)^4}$

s $y = \sqrt[3]{(7-3x)^2}$ **t** $y = \dfrac{5}{\sqrt{4+x}}$ **u** $y = \dfrac{1}{2\sqrt{3x-1}}$

v $y = \dfrac{3}{4(2x+7)^9}$ **w** $y = \dfrac{1}{x^4-3x^3+3x}$ **x** $y = \sqrt[3]{(4x+1)^4}$

y $y = \dfrac{1}{\sqrt[4]{(7-x)^5}}$

2 Find the gradient of the tangent to the curve $y = (3x-2)^3$ at the point (1, 1).

3 If $f(x) = 2(x^2-3)^5$, evaluate $f'(2)$.

4 The curve $y = \sqrt{x-3}$ has a tangent with gradient $\dfrac{1}{2}$ at point N. Find the coordinates of N.

5 For what values of x does the function $f(x) = \dfrac{1}{4x-1}$ have $f'(x) = -\dfrac{4}{49}$?

6 Find the equation of the tangent to $y = (2x+1)^4$ at the point where $x = -1$.

7 Find the equation of the tangent to the curve $y = (2x-1)^8$ at the point where $x = 1$.

8 Find the equation of the normal to the curve $y = (3x-4)^3$ at $(1, -1)$.

9 Find the equation of the normal to the curve $y = (x^2+1)^4$ at (1, 16).

10 Find the equation of **a** the tangent and **b** the normal to the curve $f(x) = \dfrac{1}{2x+3}$ at the point where $x = -1$.

WS

Product rule

8.08 Product rule

The **product rule** is a method for differentiating the product of 2 functions.

The product rule

If $y = uv$ where u and v are functions, then:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

or $\quad y' = u'v + v'u.$

We can also write the product rule the other way round (differentiating *v* first), but the above formulas will also help us to remember the quotient rule in the next section.

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EXAMPLE 16

Differentiate:

a $y = (3x + 1)(x - 5)$ **b** $y = 9x^3(2x - 7)$

Solution

a You could expand the brackets and then differentiate:

$$\begin{aligned} y &= (3x + 1)(x - 5) \\ &= 3x^2 - 15x + x - 5 \\ &= 3x^2 - 14x - 5 \end{aligned}$$

$$\frac{dy}{dx} = 6x - 14$$

Using the product rule:

$y = uv$ where $u = 3x + 1$ and $v = x - 5$

$u' = 3$ $v' = 1$

$$\begin{aligned} y' &= u'v + v'u \\ &= 3(x - 5) + 1(3x + 1) \\ &= 3x - 15 + 3x + 1 \\ &= 6x - 14 \end{aligned}$$

b $y = uv$ where $u = 9x^3$ and $v = 2x - 7$

$u' = 27x^2$ $v' = 2$

$$\begin{aligned} y' &= u'v + v'u \\ &= 27x^2(2x - 7) + 2(9x^3) \\ &= 54x^3 - 189x^2 + 18x^3 \\ &= 72x^3 - 189x^2 \end{aligned}$$

We can use the product rule together with the chain rule.

EXAMPLE 17

Differentiate:

a $y = 2x^5(5x+3)^3$ **b** $y = (3x-4)\sqrt{5-2x}$

Solution

a $y = uv$ where $u = 2x^5$ and $v = (5x+3)^3$

$u' = 10x^4$ $\quad v' = 5 \times 3(5x+3)^2$ using chain rule

$= 15(5x+3)^2$

$$\begin{aligned} y' &= u'v + v'u \\ &= 10x^4\,(5x+3)^3 + 15(5x+3)^2\,2x^5 \\ &= 10x^4(5x+3)^3 + 30x^5(5x+3)^2 \\ &= 10x^4(5x+3)^2[(5x+3)+3x] \\ &= 10x^4(5x+3)^2(8x+3) \end{aligned}$$

b $y = uv$ where $u = 3x-4$ and $v = \sqrt{5-2x} = (5-2x)^{\frac{1}{2}}$

$u' = 3$ $\quad v' = -2 \times \frac{1}{2}(5-2x)^{-\frac{1}{2}}$ using chain rule

$$\begin{aligned} &= -(5-2x)^{-\frac{1}{2}} \\ &= -\frac{1}{(5-2x)^{\frac{1}{2}}} \\ &= -\frac{1}{\sqrt{5-2x}} \end{aligned}$$

$$\begin{aligned} y' &= u'v + v'u \\ &= 3 \cdot \sqrt{5-2x} + -\frac{1}{\sqrt{5-2x}}(3x-4) \\ &= 3\sqrt{5-2x} - \frac{3x-4}{\sqrt{5-2x}} \\ &= \frac{3\sqrt{5-2x} \times \sqrt{5-2x}}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}} \\ &= \frac{3(5-2x)}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}} \end{aligned}$$

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$$= \frac{3(5-2x)-(3x-4)}{\sqrt{5-2x}}$$

$$= \frac{15-6x-3x+4}{\sqrt{5-2x}}$$

$$= \frac{19-9x}{\sqrt{5-2x}}$$

Exercise 8.08 Product rule

1 Differentiate:

a $y = x^3(2x + 3)$ **b** $y = (3x - 2)(2x + 1)$ **c** $y = 3x(5x + 7)$

d $y = 4x^4(3x^2 - 1)$ **e** $y = 2x(3x^4 - x)$ **f** $y = x^2(x + 1)^3$

g $y = 4x(3x - 2)^5$ **h** $y = 3x^4(4 - x)^3$ **i** $y = (x + 1)(2x + 5)^4$

2 Find the gradient of the tangent to the curve $y = 2x(3x - 2)^4$ at (1, 2).

3 If $f(x) = (2x + 3)(3x - 1)^5$, evaluate $f'(1)$.

4 Find the exact gradient of the tangent to the curve $y = x\sqrt{2x+5}$ at the point where $x = 1$.

5 Find the gradient of the tangent where $t = 3$ given $x = (2t - 5)(t + 1)^3$.

6 Find the equation of the tangent to the curve $y = x^2(2x - 1)^4$ at (1, 1).

7 Find the equation of the tangent to $h = (t + 1)^2(t - 1)^7$ at (2, 9).

8 Find exact values of x for which the gradient of the tangent to the curve $y = 2x(x + 3)^2$ is 14.

9 Given $f(x) = (4x - 1)(3x + 2)^2$, find the equation of the tangent at the point where $x = -1$.

8.09 Quotient rule

WS Quotient rule

WS Rules of differentiation

WS Mixed differentiation problems

The **quotient rule** is a method for differentiating the ratio of 2 functions.

The quotient rule

If $y = \frac{u}{v}$ where u and v are functions, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\text{or } y' = \frac{u'v - v'u}{v^2}.$$

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EXAMPLE 18

Differentiate:

a $y = \dfrac{3x - 5}{5x + 2}$

b $y = \dfrac{4x^3 - 5x + 2}{x^3 - 1}$

Solution

a $y = \dfrac{u}{v}$ where $u = 3x - 5$ and $v = 5x + 2$

$u' = 3$ $\quad v' = 5$

$$
\begin{aligned}
y' &= \frac{u'v - v'u}{v^2} \\
&= \frac{3(5x + 2) - 5(3x - 5)}{(5x + 2)^2} \\
&= \frac{15x + 6 - 15x + 25}{(5x + 2)^2} \\
&= \frac{31}{(5x + 2)^2}
\end{aligned}
$$

b $y = \dfrac{u}{v}$ where $u = 4x^3 - 5x + 2$ and $v = x^3 - 1$

$u' = 12x^2 - 5$ $\quad v' = 3x^2$

$$
\begin{aligned}
y' &= \frac{u'v - v'u}{v^2} \\
&= \frac{(12x^2 - 5)(x^3 - 1) - 3x^2(4x^3 - 5x + 2)}{(x^3 - 1)^2} \\
&= \frac{12x^5 - 12x^2 - 5x^3 + 5 - 12x^5 + 15x^3 - 6x^2}{(x^3 - 1)^2} \\
&= \frac{10x^3 - 18x^2 + 5}{(x^3 - 1)^2}
\end{aligned}
$$

Exercise 8.09 Quotient rule

1 Differentiate:

a $y = \dfrac{1}{2x - 1}$ **b** $y = \dfrac{3x}{x + 5}$ **c** $y = \dfrac{x^3}{x^2 - 4}$ **d** $y = \dfrac{x - 3}{5x + 1}$

e $y = \dfrac{x - 7}{x^2}$ **f** $y = \dfrac{5x + 4}{x + 3}$ **g** $y = \dfrac{x}{2x^2 - 1}$ **h** $y = \dfrac{x + 4}{x - 2}$

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i $y = \frac{2x+7}{4x-3}$ **j** $y = \frac{x+5}{3x+1}$ **k** $y = \frac{x+1}{3x^2-7}$ **l** $y = \frac{2x^2}{2x-3}$

m $y = \frac{x^2+4}{x^2-5}$ **n** $y = \frac{x^3}{x+4}$ **o** $y = \frac{x^3+2x-1}{x+3}$ **p** $y = \frac{x^2-2x-1}{3x+4}$

q $y = \frac{2x}{(x+5)^{\frac{1}{2}}}$ **r** $y = \frac{x-1}{(7x+2)^4}$ **s** $y = \frac{3x+1}{\sqrt{x+1}}$ **t** $y = \frac{\sqrt{x-1}}{2x-3}$

2 Find the gradient of the tangent to the curve $y = \frac{2x}{3x+1}$ at $\left(1, \frac{1}{2}\right)$.

3 If $f(x) = \frac{4x+5}{2x-1}$, evaluate $f'(2)$.

4 Find values of x for which the gradient of the tangent to $y = \frac{4x-1}{2x-1}$ is –2.

5 Given $f(x) = \frac{2x}{x+3}$, find x if $f'(x) = \frac{1}{6}$.

6 Find the equation of the tangent to the curve $y = \frac{x}{x+2}$ at $\left(4, \frac{2}{3}\right)$.

7 Find the equation of the tangent to the curve $y = \frac{x^2-1}{x+3}$ at $x = 2$.

8.10 Rates of change

WS Rates of change

WS Instantaneous rates of change

WS Graphs of rates of change

We know that the gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$ of the secant passing through 2 points on the graph of a function gives the **average rate of change** between those 2 points.

Now consider a quantity Q that changes with time, giving the function $Q(t)$.

Average rate of change

The average rate of change of a quantity Q with respect to time t is $\frac{Q_2 - Q_1}{t_2 - t_1}$.

We know that the gradient $\frac{dy}{dx}$ of the tangent at a point on the graph of a function gives the **instantaneous rate of change** at that point.

Instantaneous rate of change

The instantaneous rate of change of a quantity Q with respect to time t is $\frac{dQ}{dt}$.

EXAMPLE 19

a The number of bacteria in a culture increases according to the function $B = 2t^4 - t^2 + 2000$, where t is time in hours. Find:

i the number of bacteria initially

ii the average rate of change in number of bacteria between 2 and 3 hours

iii the number of bacteria after 5 hours

iv the rate at which the number of bacteria is increasing after 5 hours.

b An object travels a distance according to the function $D = t^2 + t + 5$, where D is in metres and t is in seconds. Find the speed at which it is travelling at:

i 4 s **ii** 10 s

Solution

a i $B = 2t^4 - t^2 + 2000$

Initially, $t = 0$:

$$B = 2(0)^4 - (0)^2 + 2000$$
$$= 2000$$

So there are 2000 bacteria initially.

ii When $t = 2$, $B = 2(2)^4 - (2)^2 + 2000$
$$= 2028$$

When $t = 3$, $B = 2(3)^4 - (3)^2 + 2000$
$$= 2153$$

$$\text{Average rate of change} = \frac{B_2 - B_1}{t_2 - t_1}$$
$$= \frac{2153 - 2028}{3 - 2}$$
$$= 125 \text{ bacteria/hour}$$

So the average rate of change is 125 bacteria per hour.

iii When $t = 5$, $B = 2(5)^4 - (5)^2 + 2000$
$$= 3225$$

So there will be 3225 bacteria after 5 hours.

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iv The instantaneous rate of change is given by the derivative $\frac{dB}{dt} = 8t^3 - 2t$.

When $t = 5$, $\frac{dB}{dt} = 8(5)^3 - 2(5)$

$= 990$

So the rate of increase after 5 hours will be 990 bacteria per hour.

b Speed is the rate of change of distance over time: $\frac{dD}{dt} = 2t + 1$.

i When $t = 4$, $\frac{dD}{dt} = 2(4) + 1$

$= 9$

So speed after 4 s is 9 m/s.

ii When $t = 10$, $\frac{dD}{dt} = 2(10) + 1$

$= 21$

So speed after 10 s is 21 m/s.

Displacement, velocity and acceleration

Displacement (x) measures the distance of an object from a fixed point (origin). It can be positive or negative or 0, according to where the object is.

Velocity (v) is the rate of change of displacement with respect to time, and involves speed and direction.

Velocity

Velocity $v = \frac{dx}{dt}$ is the instantaneous rate of change of displacement x over time t.

Acceleration (a) is the rate of change of velocity with respect to time.

Acceleration

Acceleration $a = \frac{dv}{dt}$ is the instantaneous rate of change of velocity v over time t.

We usually write velocity units as km/h or m/s, but we can also use index notation and write km h^{-1} or m s^{-1}.

With acceleration units, we write km/h/h as km/h^2, or in index notation we write km h^{-2}.

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EXAMPLE 20

A ball rolls down a ramp so that its displacement x cm in t seconds is $x = 16 - t^2$.

a Find its initial displacement.

b Find its displacement at 3 s.

c Find its velocity at 2 s.

d Show that the ball has a constant acceleration of -2 cm s^{-2}.

Solution

a $x = 16 - t^2$

Initially, $t = 0$:

$x = 16 - 0^2$

$= 16$

So the ball's initial displacement is 16 cm.

b When $t = 3$:

$x = 16 - 3^2$

$= 7$

So the ball's displacement at 3 s is 7 cm.

c $v = \frac{dx}{dt}$

$= -2t$

When $t = 2$:

$v = -2(2)$

$= -4$

So the ball's velocity at 2 s is -4 cm s^{-1}.

x is measured in cm, t is measured in s, so v is measured in cm/s or cm s^{-1}.

d $a = \frac{dv}{dt}$

$= -2$

So acceleration is constant at -2 cm s^{-2}.

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Exercise 8.10 Rates of change

1 Find the formula for the rate of change for each function.

a $h = 20t - 4t^2$ **b** $D = 5t^3 + 2t^2 + 1$ **c** $A = 16x - 2x^2$

d $x = 3t^5 - t^4 + 2t - 3$ **e** $V = \frac{4}{3}\pi r^3$ **f** $S = 2\pi r + \frac{50}{r^2}$

g $D = \sqrt{x^2 - 4}$ **h** $S = 800r + \frac{400}{r}$

2 If $h = t^3 - 7t + 5$, find:

a the average rate of change of h between $t = 3$ and $t = 4$

b the instantaneous rate of change of h when $t = 3$.

3 The volume of water V in litres flowing through a pipe after t seconds is given by $V = t^2 + 3t$. Find the rate at which the water is flowing when $t = 5$.

4 The mass in grams of a melting ice block is given by the formula $M = t - 2t^2 + 100$, where t is time in minutes.

a Find the average rate of change at which the ice block is melting between:

i 1 and 3 minutes **ii** 2 and 5 minutes.

b Find the rate at which it will be melting at 5 minutes.

5 The surface area in cm^2 of a balloon being inflated is given by $S = t^3 - 2t^2 + 5t + 2$, where t is time in seconds. Find the rate of increase in the balloon's surface area at 8 s.

6 A circular disc expands as it is heated. The area, in cm^2, of the disc increases according to the formula $A = 4t^2 + t$, where t is time in minutes. Find the rate of increase in the area after 5 minutes.

7 A car is d km from home after t hours according to the formula $d = 10t^2 + 5t + 11$.

a How far is the car from home:

i initially? **ii** after 3 hours? **iii** after 5 hours?

b At what speed is the car travelling after:

i 3 hours? **ii** 5 hours?

8 According to Boyle's Law, the pressure of a gas is given by the formula $P = \frac{k}{V}$ where k is a constant and V is the volume of the gas. If $k = 100$ for a certain gas, find the rate of change in the pressure when $V = 20$.

9 The displacement of a particle is $x = t^3 - 9t$ cm, where t is time in seconds.

a Find the velocity of the particle at 3 s.

b Find the acceleration at 2 s.

c Show that the particle is initially at the origin, and find any other times that the particle will be at the origin.

d At what time will the acceleration be 30 cm s^{-2}?

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10 A particle is moving with displacement $s = 2t^2 - 8t + 3$, where s is in metres and t is in seconds.

a Find its initial velocity.

b Show that its acceleration is constant and find its value.

c Find its displacement at 5 s.

d Find when the particle's velocity is zero.

e What will the particle's displacement be at that time?

WS

Related rates of change

EXT1 8.11 Related rates of change

Rates of change with respect to time are harder to calculate when there are **2 or more related variables**. For example, when inflating a balloon, both its radius and its volume increase, but at different rates.

Related rates of change

If y is related to x and x is related to time t, then the instantaneous rate of change of y with respect to t uses the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

EXAMPLE 21

Related rates of change

a Given $y = 2x^2 - 3x + 1$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = -2$.

b A spherical metal ball is heated so that its radius is expanding at the rate of 0.04 mm per second. At what rate will its volume be increasing when the radius is 3.4 mm?

c A pool holds a volume of water given by $V = 2x + 3x^2$, where x is the depth of water. If the pool is filled with water at the rate of 1.3 m^3/h, at what rate will the level of water be increasing when the depth is 0.78 m?

d Car A is north of an intersection and travelling towards it, while car B is moving away from the intersection eastwards at a constant speed of 60 km h^{-1}. The distance between the cars at any one time is 10 km. Find the rate at which car A will be moving when car B is 8 km from the intersection.

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Solution

a $y = 2x^2 - 3x + 1,$

so $\frac{dy}{dx} = 4x - 3.$

Also, $\frac{dx}{dt} = 5.$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= (4x - 3) \times 5$$

$$= 20x - 15$$

When $x = -2$,

$$\frac{dy}{dt} = 20(-2) - 15$$

$$= -55$$

b $V = \frac{4}{3}\pi r^3$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

Also, $\frac{dr}{dt} = 0.04$ mm/s

Now $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$$= 4\pi r^2 \times 0.04$$

$$= 0.16\pi r^2$$

When $r = 3.4$, $\frac{dV}{dt} = 0.16\pi(3.4)^2$

$$\approx 5.81$$

So the volume is increasing at a rate of 5.81 mm^3/s.

c $V = 2x + 3x^2$

$$\therefore \frac{dV}{dx} = 2 + 6x$$

Also, $\frac{dV}{dt} = 1.3$ m^3/h

Now $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$$1.3 = (2 + 6x) \times \frac{dx}{dt}$$

When $x = 0.78$,

$$1.3 = [2 + 6(0.78)] \times \frac{dx}{dt}$$

$$= 6.68\frac{dx}{dt}$$

$$0.195 \approx \frac{dx}{dt}$$

So the level of water is increasing at a rate of 0.195 m/h.

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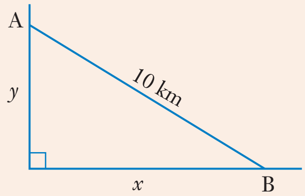
d $x^2 + y^2 = 100$ (by Pythagoras' theorem)

$y^2 = 100 - x^2$

$\therefore y = \sqrt{100 - x^2}$

$= \left(100 - x^2\right)^{\frac{1}{2}}$



$$\frac{dy}{dx} = \frac{1}{2}\left(100 - x^2\right)^{-\frac{1}{2}}(-2x)$$

$$= -x\left(100 - x^2\right)^{-\frac{1}{2}}$$

$$= \frac{-x}{\sqrt{100 - x^2}}$$

$\dfrac{dx}{dt} = -60$ (the speed of car B)

Note: The negative sign means the car is moving **away** from the intersection.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{-x}{\sqrt{100 - x^2}} \times (-60)$$

$$= \frac{60x}{\sqrt{100 - x^2}}$$

When $x = 8$,

$$\frac{dy}{dt} = \frac{60(8)}{\sqrt{100 - 8^2}}$$

$$= \frac{480}{6}$$

$$= 80$$

So car A will be travelling at a speed of 80 km h^{-1} when car B is 8 km from the intersection.

EXT1 Exercise 8.11 Rates involving two variables

1 Find an expression for $\dfrac{dy}{dt}$ given:

a $y = x^4$ and $\dfrac{dx}{dt} = 2$

b $y = 3x^3 + 7$ and $\dfrac{dx}{dt} = 6$

c $y = x^2 - x - 2$ and $\dfrac{dx}{dt} = -3$

2 Evaluate $\dfrac{dy}{dt}$ when $x = 4$, given:

a $y = 2x^3 + 3x - 7$ and $\dfrac{dx}{dt} = 3$

b $y = (3x + 1)^3$ and $\dfrac{dx}{dt} = -4$

c $y = (5 - x)^5$ and $\dfrac{dx}{dt} = 4$

3 If $y = x^3 + 5x - 4$ and $\dfrac{dy}{dt} = 6$, evaluate $\dfrac{dx}{dt}$ when $x = 2$.

4 If $y = x^2 + x$ and $\dfrac{dy}{dt} = -5$, evaluate $\dfrac{dx}{dt}$ when $x = 4$.

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5 If $x = 3t^2 - t$ and $\frac{dt}{du} = 6$, evaluate $\frac{dx}{du}$ when $t = 12$.

6 If $V = 5\pi r^2$ and $\frac{dr}{dt} = 2$, evaluate $\frac{dV}{dt}$ correct to the nearest unit when $r = 4.6$.

7 If $A = 16x - 2x^2$ and $\frac{dx}{dt} = 11$, evaluate $\frac{dA}{dt}$ when $x = 3$.

8 If $V = x^3 + 4x^2 - 3x + 4$ and $\frac{dV}{dt} = 10$, evaluate $\frac{dx}{dt}$ when $x = 1$.

9 A cube is expanding so that its side length is increasing at the constant rate of 0.12 mm s^{-1}. Find the rate of increase in its volume when its side is 150 mm.

10 The radius of a cylindrical pipe 2 m long expands with heat at a constant rate of 1.2×10^{-3} mm s^{-1}. Find the rate at which the volume of the pipe will be increasing when its radius is 19 mm.

11 Find the rate of change of the surface area of a balloon when its radius is 6.3 cm, if the radius is expanding at a constant rate of 1.3 cm s^{-1}.

12 A cone contains liquid with volume given by $V = \frac{6\pi h^2}{7}$, where h is the height of the liquid in the cone. If the height of the liquid is increasing at a rate of 2.3 cm s^{-1}, find the rate of increase in the volume of the liquid when its height is 12.9 cm.

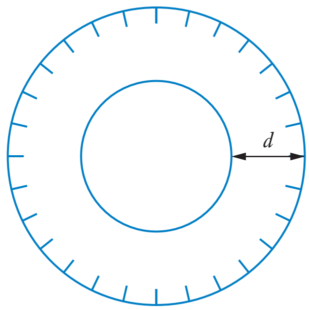
13 A point, P, moves along the curve $y = 2x^2 - 7x + 9$. What will be the rate of change in the y-coordinate of P when the x-coordinate is increasing at a rate of 8 units per second and the value of x is 3?

14 An ice cube with sides x mm is melting so that the length of its sides is decreasing at 0.8 mm s^{-1}. What will the rate of decrease in volume be when the sides are 120 mm long?

15 A factory produces a quantity of radios according to the formula $N = x^2 + 7x$, where x is the number of workers. If the number of workers decreases by a constant rate of 2 per week, find the rate at which the quantity of radios made will decrease when there are 150 workers.

16 A particle is moving so that its velocity is given by the formula $v = 8x^3 - 5x^2 - 3x - 1$, where x is its displacement. If the rate of change in displacement is a constant 4.2 cm s^{-1}, find the rate of change in velocity when the displacement is –4.7 cm.

17 A car tyre has a volume given by the formula $V = 0.53\pi d^2$, where d is the diameter of the wall of the tyre. If the diameter decreases at the constant rate of 0.02 mm s^{-1}, find the rate at which the volume of the tyre will be decreasing when the diameter is 167 mm.



18 The number of burrows for a colony of rabbits is decreasing owing to the clearing of land, at a constant rate of 5 burrows per day. If the number of rabbits is given by the formula $N = 5x^2 + 3x$, where x is the number of burrows, find the rate of decrease in the rabbit population when there are 55 burrows.

19 The volume of a balloon being inflated is increasing at a constant rate of 115 cm^3 s^{-1}. Find the rate of increase in its radius when the radius is 3 cm.

20 A population increases at a constant rate of 15 000 people per year. If the population has the formula $P = x^2 - 3000x + 100$, where x is the number of houses available, find the rate at which the number of houses will be increasing when there are 5000 houses.

21 A cone-shaped candle whose height is 3 times its radius is melting at the constant rate of 1.4 cm^3 s^{-1}. If the proportion of radius to height is preserved, find the rate at which the radius will be decreasing when it is 3.7 cm.

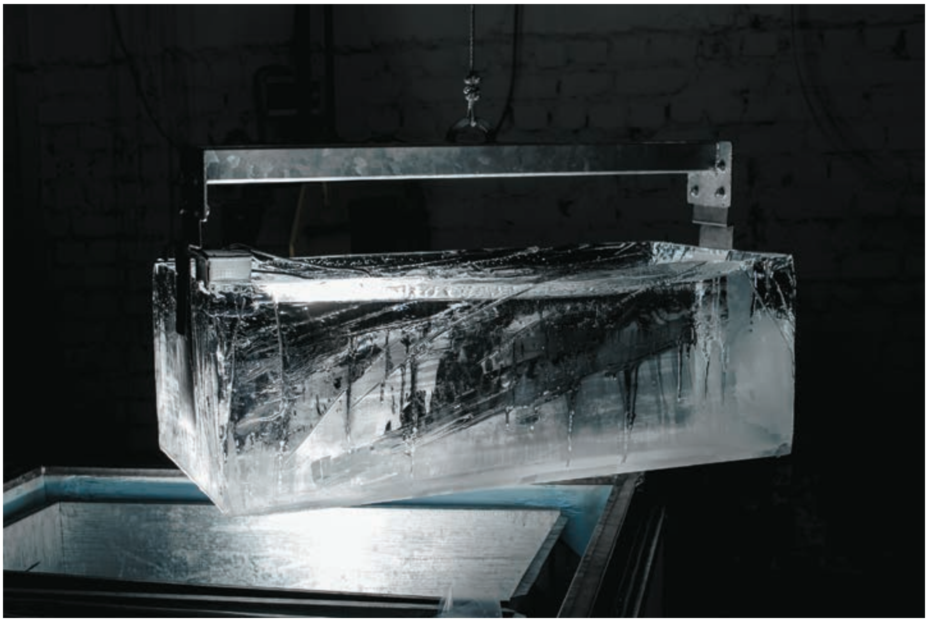
22 The rate of change in the radius of a sphere is 0.3 mm s^{-1}. Given that the radius is 88 mm at a certain time, find the rate of change at that time in:

a surface area **b** volume

23 If the volume of a cube is increasing at the rate of 23 mm^3 s^{-1}, find the increase in its surface area when its side length is 140 mm.

24 The surface area of a spherical bubble is increasing at a constant rate of 1.9 mm^2 s^{-1}. Find the rate of increase in its volume when its radius is 0.6 mm.

25 A rectangular block of ice with a square base has a height half the side of the base. As it melts, the volume of the block of ice is decreasing at the rate of 12 cm^3 s^{-1}. Find the rate at which its surface area will be decreasing when the side of its base is 2.1 cm.



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EXT1 8.12 Motion in a straight line

WS Motion in a straight line

WS Straight-line motion 1

Now we will look more closely at the rate of change of motion along a straight line.

When studying the motion of a particle, the term 'particle' describes any moving object, such as a golf ball or a car. We will also ignore friction, gravity and other influences on the motion.

DID YOU KNOW?

The origins of calculus

Calculus was developed in the 17th century as a solution to problems about the **study of motion**. Some problems of the time included finding the speed and acceleration of planets, calculating the lengths of their orbits, finding maximum and minimum values of functions, finding the direction in which an object is moving at any one time and calculating the areas and volumes of certain figures.

Displacement

Displacement (x) measures the distance of a particle from a fixed point (origin). Displacement can be positive or negative, according to which side of the origin, O, it is on. Usually, it is **positive** to the **right** of O and **negative** to the **left** of O.

Zero displacement

When the particle is at the **origin**, its **displacement** is **zero**. That is, $x = 0$.

Velocity

Velocity (v) is the rate of change of displacement:

Velocity

$$v = \dot{x} = \frac{dx}{dt}$$

$\dot{x}$ is another way of writing $\frac{dx}{dt}$.

If the particle is moving to the **right**, velocity is **positive**. If it is moving to the **left**, velocity is **negative**.

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Zero velocity

When the particle is not moving, we say that it is **at rest**. That is, $\dot{x} = 0$.

Acceleration

Acceleration is the rate of change of velocity:

Acceleration

$$a = \ddot{x} = \frac{dv}{dt}$$

$\ddot{x}$ means the derivative of the derivative of x, or the second derivative of x. If the acceleration (and the force on the particle) is to the **right**, it is **positive**. If the acceleration is to the **left**, it is **negative**.

If the acceleration is in the **same direction** as the velocity, the particle is **speeding up** (accelerating). If the acceleration is in the **opposite direction** from the velocity, the particle is **slowing down** (decelerating). If the particle is not speeding up or slowing down, then its speed is not changing and we say it has constant velocity.

Zero acceleration

When the particle is not accelerating, we say that it has **constant velocity**. That is, $\ddot{x} = 0$.

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Motion graphs

We can describe the velocity of a particle by looking at the derivative function of its displacement graph.

We can describe acceleration by looking at the derivative function of its velocity graph.

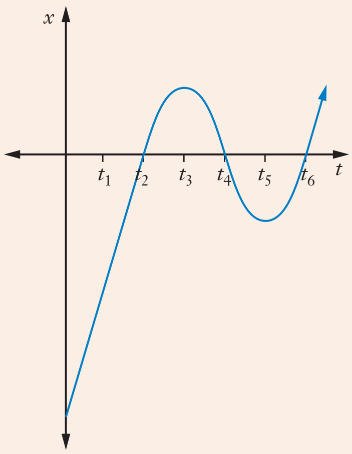
EXAMPLE 22

a This graph shows the displacement of a particle from the origin as it moves in a straight line.

i When is the particle at rest?

ii When is it at the origin?

iii When is it moving at its greatest speed?



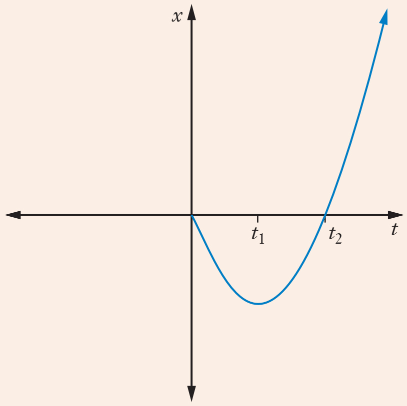
b This graph shows the displacement x of a particle over time t.

i Sketch a graph of its velocity.

ii Sketch its acceleration graph.

iii Find when the particle is at the origin.

iv Find when the particle is at rest.



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Solution

a **i** The particle is at rest when $v = \dfrac{dx}{dt} = 0$.

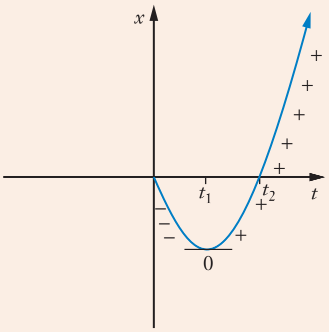
This is where the displacement is neither increasing nor decreasing.

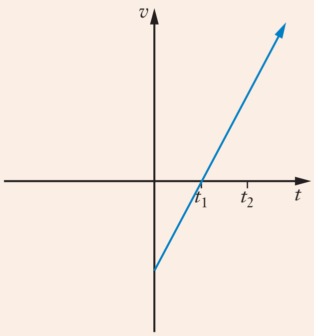
From the graph, the particle is at rest at times t_3 and t_5.

ii The particle is at the origin when $x = 0$, that is, on the t-axis.
So the particle is at the origin at times t_2, t_4 and t_6.

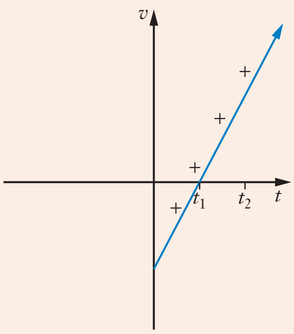
iii The speed is greatest when the graph is at its steepest. From the graph, this occurs at t_4. The particle is moving at its greatest speed at time t_4.

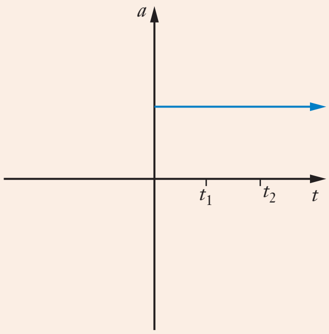
b **i** Velocity $v = \dfrac{dx}{dt}$. By noting where the gradient of the displacement graph is positive, negative and zero, we draw its gradient function for the velocity graph.





ii Acceleration $a = \dfrac{dv}{dt}$. By noting where the gradient of the velocity graph is positive, negative and zero, we draw its gradient function for the acceleration graph.





iii At the origin, $x = 0$.

From the displacement graph, this is at 0 and t_2.

iv At rest, $v = 0$.

From the velocity graph, this is at t_1.

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Motion functions

EXAMPLE 23

a The displacement x cm of a particle at t seconds is given by $x = 4t - t^2$.

i Find when the particle is at rest.

ii How far does the particle move in the first 3 seconds?

b The displacement of a particle is given by $x = -t^2 + 2t + 3$ cm where t is in seconds.

i Find the initial velocity of the particle.

ii Show that the particle has constant acceleration.

iii Find when the particle is at the origin.

iv Find the particle's greatest displacement.

v Sketch the graph showing the particle's motion.

Solution

a **i** At rest, $v = 0$.

$$v = \frac{dx}{dt} = 4 - 2t$$

$$4 - 2t = 0$$

$$4 = 2t$$

$$2 = t$$

So the particle is at rest at 2 seconds.

ii When $t = 0$:

$$x = 4(0) - 0^2$$
$$= 0$$

So the particle is initially at the origin.

When $t = 3$:

$$x = 4(3) - 3^2$$
$$= 3$$

So the particle is 3 cm from the origin at 3 seconds.

When $t = 2$:

$$x = 4(2) - 2^2$$
$$= 4$$

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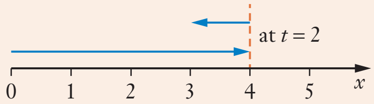
The particle moves from $x = 0$ to 4 in the first 2 seconds, then it turns and moves to $x = 3$ in the 3rd second, going back 1 cm (see diagram).



Total distance travelled $= 4 + 1$

$= 5$ cm

b **i** $v = \dfrac{dx}{dt} = -2t + 2$

Initially, $t = 0$:

$v = -2(0) + 2$

$= 2$

So the initial velocity is 2 cm s^{-1}.

ii $v = -2t + 2$

Acceleration $a = \dfrac{dv}{dt} = -2$

$\therefore$ the particle has a constant acceleration of -2 cm s^{-2}.

iii At the origin $\quad x = 0$

$$-t^2 + 2t + 3 = 0$$

$$-(t + 1)(t - 3) = 0$$

$$t = -1 \text{ or } 3$$

Since time cannot be negative, the particle will be at the origin at 3 s.

iv Greatest displacement will be at the turning point of the displacement graph (or when $v = 0$):

$$\frac{dx}{dt} = 0$$

$$-2t + 2 = 0$$

$$2 = 2t$$

$$1 = t$$

So the greatest displacement occurs when $t = 1$.

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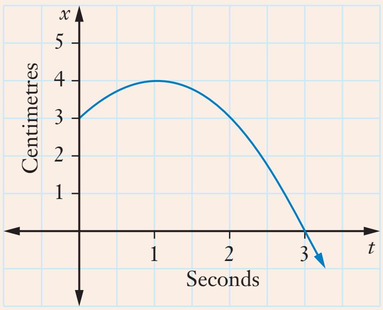
When $t = 1$:

$x = -1^2 + 2(1) + 3$

$= 4$

So the greatest displacement is 4 cm.

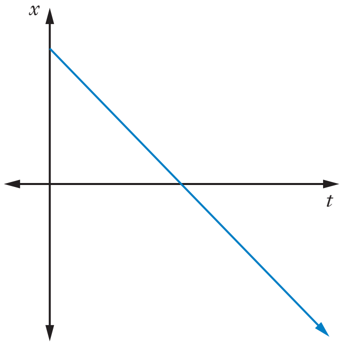
v The graph of $x = -t^2 + 2t + 3$ is a concave downward parabola. From above, we know that it has x-intercepts at -1 and 3 and goes through (0, 3) and (1, 4). We draw the graph for $t \geq 0$ only.



EXT1 Exercise 8.12 Motion in a straight line

1 For each displacement graph, sketch the graphs for velocity and acceleration.

a



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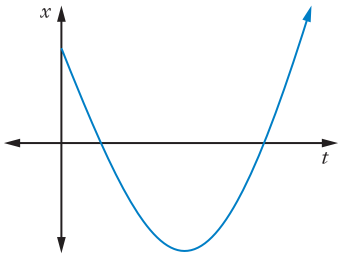

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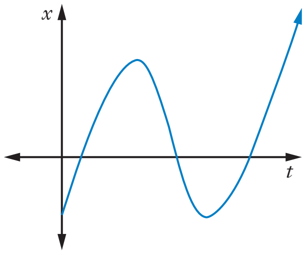

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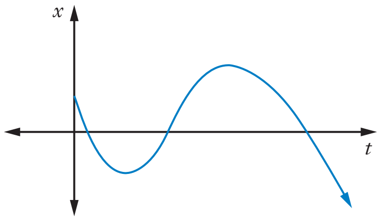

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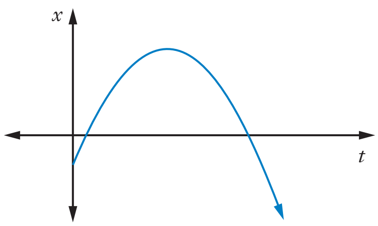

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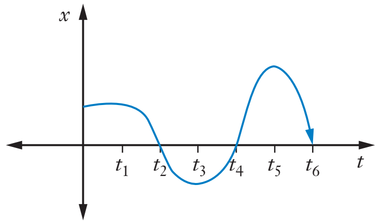
2 The graph shows the displacement of a particle as it moves along a straight line.

When is the particle:

a at the origin?

b at rest?

c furthest from the origin?



3 This graph shows the velocity of a particle.

a When is the particle at rest?

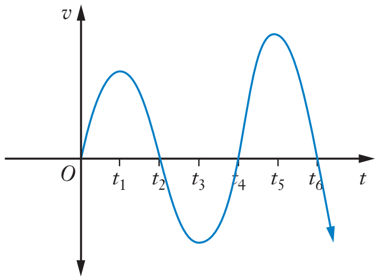
b When is the acceleration zero?

c When is the speed the greatest?

d Describe the motion of the particle at:

i t_2

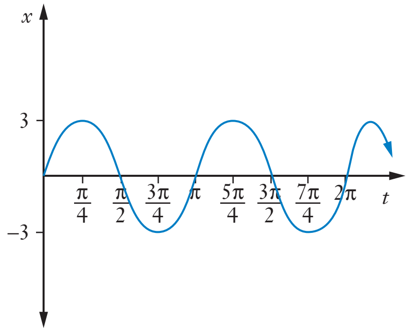
ii t_3



4 This graph shows the displacement of a pendulum.

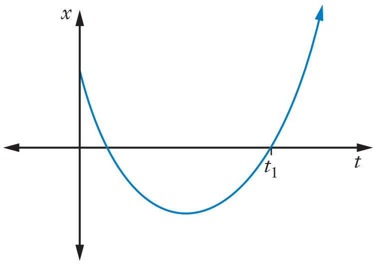
a When is the pendulum at rest?

b When is the pendulum in its equilibrium position (at the origin)?



5 Describe the displacement and velocity of the particle at t_1 for each displacement graph.

a



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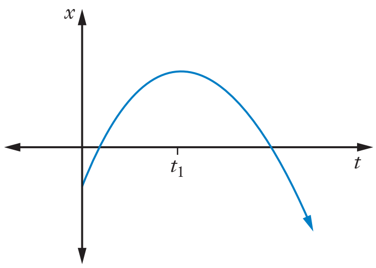

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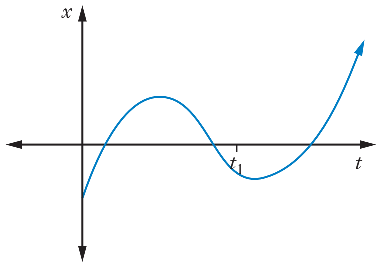

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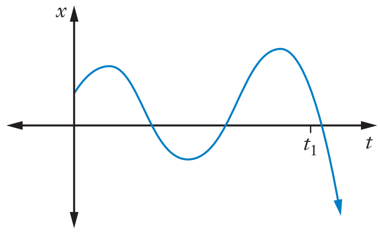

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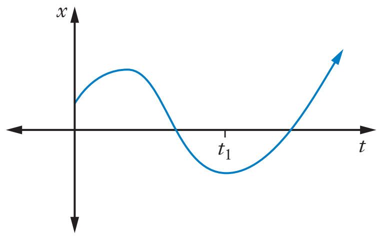

6 A projectile is fired into the air and its height in metres is given by $h = 40t - 5t^2 + 4$, where t is in seconds.

a Find the initial height.

b Find the initial velocity.

c Find the height at 1 s.

d What is the maximum height of the projectile?

e Sketch the graph of the height against time.

7 The displacement in cm after time t s of a particle moving in a straight line is given by $x = 2 - t - t^2$.

a Find the initial displacement.

b Find when the particle will be at the origin.

c Find the displacement at 2 s.

d How far will the particle move in the first 2 seconds?

e Find its velocity at 3 s.

8 An object is travelling along a straight line over time t seconds, with displacement $x = t^3 + 6t^2 - 2t + 1$ m.

a Find the equations of its velocity and acceleration.

b What will its displacement be after 5 s?

c What will its velocity be after 5 s?

d Find its acceleration after 5 s.

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9 The displacement, in centimetres, of a body is given by $x = (4t - 3)^5$ where t is time in seconds.

a Find the equations for velocity $\dot{x}$ and acceleration $\ddot{x}$.

b Find the values of x, $\dot{x}$ and $\ddot{x}$ after 1 s.

c Describe the motion of the body after 1 s.

10 The displacement of a particle, in metres, over time t seconds is $s = ut + \frac{1}{2}gt^2$ where $u = 5$ and $g = -10$.

a Find the equation of the velocity of the particle.

b Find the velocity at 10 s.

c Show that the acceleration is equal to g.

11 The displacement in metres after t seconds is given by $s = \frac{2t-5}{3t+1}$. Find the equations for velocity and acceleration.

12 The displacement of a particle is given by $x = t^3 - 4t^2 + 3t$ where x is in metres and t is in seconds.

a Find the initial velocity.

b Find the times when the particle will be at the origin.

c Find the acceleration after 3 s.

13 The height of a projectile is given by $h = 7 + 6t - t^2$ where height is in metres and time is in seconds.

a Find the initial height.

b Find the maximum height reached.

c When will the projectile reach the ground?

d Sketch the graph showing the height of the projectile over time t.

e How far will the projectile travel in the first 4 s?

14 A ball is rolled up a slope at a distance from the base of the slope, after time t seconds, given by $x = 15t - 3t^2$ metres.

a How far up the slope will the ball roll before it starts to roll back down?

b What will its velocity be when it reaches the base of the slope?

c How long will the motion of the ball take altogether?

15 The displacement of a particle is given by $x = 2t^3 - 3t^2 + 42t$.

a Show that the particle is initially at the origin but never returns to the origin.

b Show that the particle is never at rest.

16 A particle is moving in a straight line so that its displacement x cm over time t seconds is given by $x = t\sqrt{49 - t^2}$.

a For how many seconds does the particle travel?

b Find the exact time at which the particle comes to rest.

c How far does the particle move altogether?

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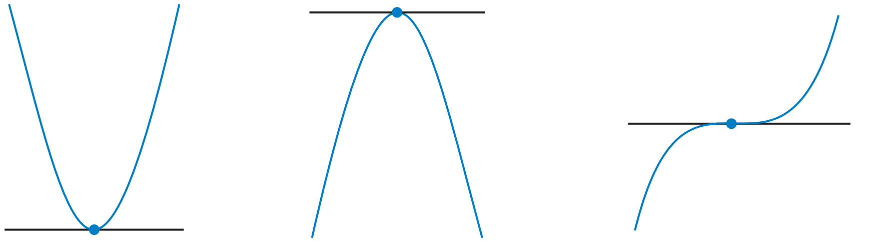
EXT1 8.13 Multiple roots of polynomial equations

WS Multiple roots

We saw in Chapter 6, *Polynomials and inverse functions*, that there is always a stationary point at a multiple root of a polynomial equation $P(x) = 0$.



As stationary points have a horizontal tangent, $P'(x) = 0$ for multiple roots.

Multiple roots of polynomial equations

If $P(x) = 0$ has a multiple root at $x = k$, then $P(k) = P'(k) = 0$.

Proof

$$P(x) = (x - k)^r Q(x) \text{ where } Q(x) \text{ is another polynomial}$$

$$\begin{aligned}
P(k) &= (k - k)^r Q(k) \\
&= 0^r Q(0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
P'(x) &= u'v + v'u \\
&= r(x - k)^{r-1} Q(x) + Q'(x)(x - k)^r \\
P'(k) &= r(k - k)^{r-1} Q(k) + Q'(k)(k - k)^r \\
&= r(0)^{r-1} Q(k) + Q'(k)(0)^r \\
&= 0
\end{aligned}$$

So $P(k) = P'(k) = 0$

Stationary points on polynomial graphs

If the multiplicity r of a root is even, there is a maximum or minimum turning point at the multiple root.

If the multiplicity r of a root is odd, there is a horizontal point of inflection at the multiple root.

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EXAMPLE 24

A polynomial has a double root at $x = 5$.

a Write an expression for the polynomial.

b Prove that $P(5) = P'(5) = 0$.

Solution

a If $P(x)$ has a double root at $x = 5$, then $(x - 5)^2$ is a factor.

So $P(x) = (x - 5)^2 Q(x)$

b
$$\begin{aligned} P(5) &= (5 - 5)^2 Q(5) \\ &= 0^2 Q(5) \\ &= 0 \end{aligned}$$

$$\begin{aligned} P'(x) &= u'v + v'u \text{ where } u = (x - 5)^2 \text{ and } v = Q(x) \\ &= 2(x - 5)\,Q(x) + Q'(x)(x - 5)^2 \end{aligned}$$

$$\begin{aligned} P'(5) &= 2(5 - 5)\,Q(5) + Q'(5)(5 - 5)^2 \\ &= 2(0)\,Q(5) + Q'(5)\,0^2 \\ &= 0 \end{aligned}$$

So $P(5) = P'(5) = 0$

Multiplicity of roots of *P(x)* and *P′(x)*

If $P(x) = 0$ has a root at $x = k$ of multiplicity $r > 1$, then $P'(x) = 0$ has a root of multiplicity $r - 1$.

Proof

If $P(x) = 0$ has a root of multiplicity r then we can write:

$$P(x) = (x - k)^r Q(x)$$

$$\begin{aligned} P'(x) &= u'v + v'u \\ &= r(x - k)^{r-1} Q(x) + Q'(x)(x - k)^r \\ &= (x - k)^{r-1}[rQ(x) + Q'(x)(x - k)] \\ &= (x - k)^{r-1} R(x) \end{aligned}$$

So $P'(x) = 0$ has a root of multiplicity $r - 1$.

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EXAMPLE 25

If a polynomial $P(x) = 0$ has a root of multiplicity 4, show that $P'(x) = 0$ has a root of multiplicity 3.

Solution

If $P(x) = 0$ has a root of multiplicity 4 then we can write:

$P(x) = (x - k)^4 Q(x)$

$P'(x) = u'v + v'u$

where $u = (x - k)^4$ and $v = Q(x)$

$= 4(x - k)^3 Q(x) + Q'(x)(x - k)^4$

$= (x - k)^3 [4Q(x) + Q'(x)(x - k)]$

$= (x - k)^3 R(x)$

So $P'(x) = 0$ has a root of multiplicity 3.

EXT1 Exercise 8.13 Multiple roots of polynomial equations

1 $P(x) = x^3 - 7x^2 + 8x + 16$ has a double root at $x = 4$.

- **a** Show that $(x - 4)^2$ is a factor of $P(x)$.
- **b** Write $P(x)$ as a product of its factors.
- **c** Prove $P(4) = P'(4) = 0$.

2 $f(x) = x^4 + 7x^3 + 9x^2 - 27x - 54$ has a triple root at $x = -3$.

- **a** Show that $(x + 3)^3$ is a factor of $f(x)$.
- **b** Write $f(x)$ as a product of its factors.
- **c** Prove $f(-3) = f'(-3) = 0$.

3 A polynomial has a triple root at $x = k$.

- **a** Write an expression for the polynomial.
- **b** Prove that $P(k) = P'(k) = 0$.

4
- **a** Write $P(x) = x^3 + x^2 - 8x - 12$ as a product of its factors.
- **b** Find the roots of $P(x) = 0$ and state the multiplicity of each root.
- **c** For each multiple root a, prove that $P(a) = P'(a) = 0$.

5
- **a** Write $P(x) = x^5 - 2x^4 + x^3$ as a product of its factors.
- **b** Find the roots of $P(x) = 0$ and state the multiplicity of each root.
- **c** For each multiple root a, prove that $P(a) = P'(a) = 0$.

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6 A polynomial equation $P(x) = 0$ has a triple root at $x = 5$. Show that $P'(x) = 0$ has a double root at $x = 5$.

7 $P(x) = 0$ has a root of multiplicity 6 at $x = -3$. Show that $P'(x) = 0$ has a root of multiplicity 5 at $x = -3$.

8 $P(x) = 0$ has a root of multiplicity n at $x = p$. Show that $P'(x) = 0$ has a root of multiplicity $n - 1$ at $x = p$.

9 **a** Divide $P(x) = x^3 + 5x^2 + 3x - 9$ by $x^2 + 6x + 9$.

b What root of $P(x) = 0$ has multiplicity 2?

c What is the multiplicity of this root for the polynomial equation $P'(x) = 0$?

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8. TEST YOURSELF

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

Derivatives find-a-word

1 Find the derivative of $\frac{2}{3x^4}$.

A $\frac{8}{3x^5}$ **B** $-\frac{8}{3x^3}$ **C** $-\frac{8}{3x^5}$ **D** $\frac{8}{3x^3}$

2 Differentiate $3x(x^3 - 5)$.

A $4x^3$ **B** $12x^3 - 15$ **C** $9x^2$ **D** $3x^4 - 15x$

3 The derivative of $y = f(x)$ is given by:

A $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{x - h}$ **B** $\lim\limits_{x \to 0} \frac{f(x+h) - f(x)}{x}$

C $\lim\limits_{h \to 0} \frac{f(x) - f(x+h)}{h}$ **D** $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

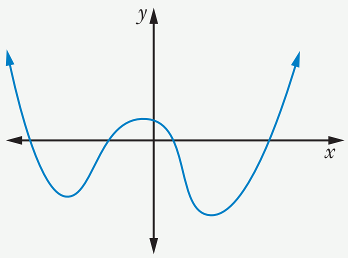
4 Which of the following is the chain rule (there is more than one answer)?

A $\frac{dy}{dx} = \frac{dy}{du} \times \frac{dx}{du}$ **B** $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

C $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$ **D** $\frac{dy}{dx} = nf(x)^{n-1}$

5 Sketch the derivative function of each graph.

a



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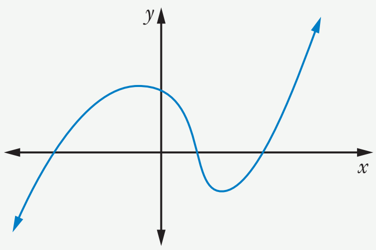

6 Differentiate $y = 5x^2 - 3x + 2$ from first principles.

7 Differentiate:

a $y = 7x^6 - 3x^3 + x^2 - 8x - 4$ **b** $y = 3x^{-4}$ **c** $y = \frac{2}{(x+1)^4}$

d $y = x^2\sqrt{x}$ **e** $y = (x^2 + 4x - 2)^9$ **f** $y = \frac{3x - 2}{2x + 1}$

g $y = x^3(3x + 1)^6$

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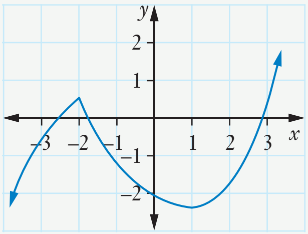
8 Find $\frac{dv}{dt}$ if $v = 2t^2 - 3t - 4$.

9 Find the gradient of the tangent to the curve $y = x^3 + 3x^2 + x - 5$ at $(1, 0)$.

10 If $h = 60t - 3t^2$, find $\frac{dh}{dt}$ when $t = 3$.

11 For each graph of a function, find all values of x where it is not differentiable.

a



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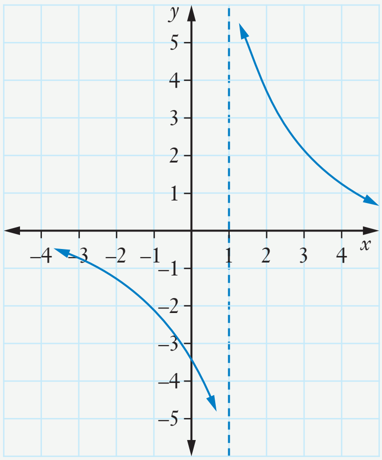

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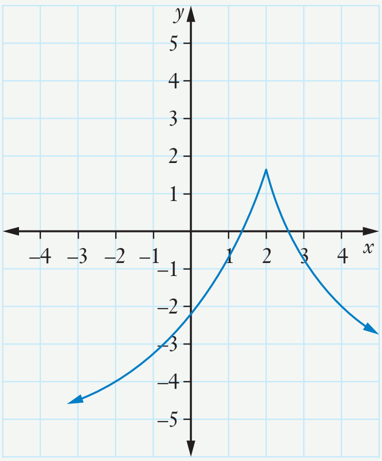

12 Differentiate:

a $y = \frac{4}{x}$

b $f(x) = \sqrt[5]{x}$

c $f(x) = 2(4x + 9)^4$

d $y = (3x + 2)(x - 1)^3$

e $f(x) = \frac{x^3 - 3}{2x + 5}$

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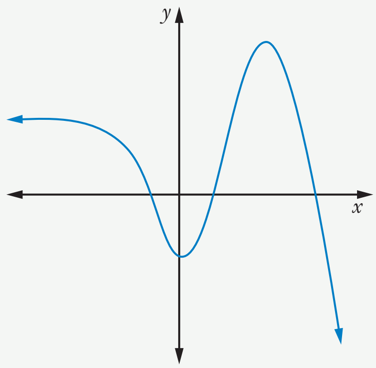
13 Sketch the derivative function of this curve.



14 Find the equation of the tangent to the curve $y = x^2 + 5x - 3$ at (2, 11).

15 Find the point on the curve $y = x^2 - x + 1$ at which the tangent has a gradient of 3.

16 Find $\frac{dS}{dr}$ if $S = 4\pi r^2$.

17 Find the gradient of the secant on the curve $f(x) = x^2 - 3x + 1$ between the points where $x = 1$ and $x = 1.1$.

18 At which points on the curve $y = 2x^3 - 9x^2 - 60x + 3$ are the tangents horizontal?

19 Find the equation of the tangent to the curve $y = x^2 + 2x - 5$ that is parallel to the line $y = 4x - 1$.

20 **a** Differentiate $s = ut + \frac{1}{2}at^2$ with respect to t.

b Find the value of t for which $\frac{ds}{dt} = 5$, $u = 7$ and $a = -10$.

21 Find the equation of the tangent to the curve $y = \frac{1}{3x}$ at the point where $x = \frac{1}{6}$.

22 A ball is thrown into the air and its height h metres over t seconds is given by $h = 4t - t^2$.

a Find the height of the ball:

i initially **ii** at 2 s

iii at 3 s **iv** at 3.5 s

b Find the average rate of change of the height between:

i 1 and 2 seconds **ii** 2 and 3 seconds

c Find the rate at which the ball is moving:

i initially **ii** at 2 s **iii** at 3 s

23 If $f(x) = x^2 - 3x + 5$, find:

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $f'(x)$

24 Given $f(x) = (4x - 3)^5$, find the value of:

a $f(1)$ **b** $f'(1)$

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25 Find $f'(4)$ when $f(x) = (x - 3)^9$.

26 Differentiate:

a $y = 3(x^2 - 6x + 1)^4$ **b** $y = \dfrac{2}{\sqrt{3x - 1}}$

27 EXT1 The volume of a sphere increases at a constant rate of 35 $\text{mm}^3\ \text{s}^{-1}$. When the radius is 12 mm, at what rate is the radius increasing?

28 The displacement x in cm of a particle over time t seconds is given by $x = 5 + 6t - 3t^2$.

a Find the initial:

i displacement **ii** velocity

b What is the velocity at 1 s?

c When is the particle at rest ($v = 0$)?

d What is the maximum displacement?

e Show that the particle is moving with constant acceleration.

29 A particle moves so that its displacement after t seconds is $x = 4t^2 - 5t^3$ metres. Find:

a its initial displacement, velocity and acceleration

b when $x = 0$

c its velocity and acceleration at 2 s.

30 EXT1 A particle has displacement $x = t^3 - 12t^2 + 36t - 9$ cm at time t seconds.

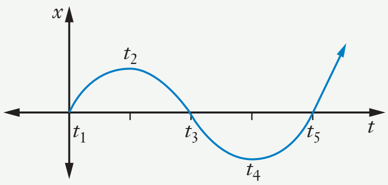
a When is the particle at rest?

b At 1 s, what is:

i the displacement? **ii** the velocity? **iii** the acceleration?

c Describe the motion of the particle after 1 s.

31 EXT1 This graph shows the displacement of a particle.



a When is the particle at the origin?

b When is it at rest?

c When is it travelling at its greatest speed?

d Sketch the graph of:

i its velocity **ii** its acceleration.

32 EXT1 The height of a ball is $h = 20t - 5t^2$ metres after t seconds.

a Find the height at 1 s.

b What is the maximum height of the ball?

c What is the time of flight of the ball?

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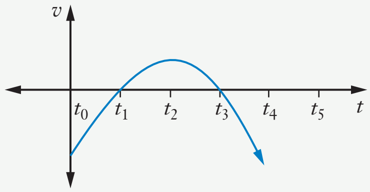
33 EXT1 This graph shows the velocity of a particle.



a Sketch a graph that shows:

i displacement **ii** acceleration.

b When is the particle at rest?

34 EXT1 $P(x) = (x - b)^7$.

a Show that $P(b) = P'(b) = 0$.

b Hence find a and b if $(x - 1)^7$ is a factor of $P(x) = x^7 + 3x^6 + ax^5 + x^4 + 3x^3 + bx^2 - x + 1$.

35 EXT1 **a** Show that $x - 5$ is a factor of $f(x) = x^3 - 7x^2 - 5x + 75$.

b Show that $f(5) = f'(5) = 0$.

c What can you say about the root at $x = 5$?

d Write $f(x)$ as a product of its factors.

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8. CHALLENGE EXERCISE

1 Find the equations of the tangents to the curve $y = x(x-1)(x+2)$ at the points where the curve cuts the x-axis.

2 **a** Find the points on the curve $y = x^3 - 6$ where the tangents are parallel to the line $y = 12x - 1$.

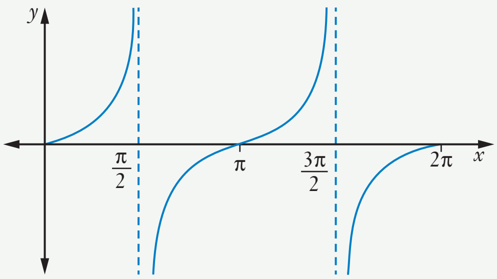
b Hence find the equations of the normals to the curve at those points.

3 The normal to the curve $y = x^2 + 1$ at the point where $x = 2$ cuts the curve again at point P. Find the coordinates of P.

4 The equation of the tangent to the curve $y = x^4 - nx^2 + 3x - 2$ at the point where $x = -2$ is given by $3x - y - 2 = 0$. Evaluate n.

5 **a** Find any points at which the graphed function is not differentiable.

b Sketch the derivative function for the graph.



6 EXT1 A sunflower grows so that the diameter of its disc increases at a constant rate of 0.1 mm per hour until it reaches 200 mm.

a Find the rate of increase in the surface area of the flower when its diameter is 50 mm.

b What is the maximum surface area of the sunflower, to the nearest mm?

7 Find the exact gradient of the tangent to the curve $y = \sqrt{x^2 - 3}$ at the point where $x = 5$.

8 Find the equation of the normal to the curve $y = 3\sqrt{x+1}$ at the point where $x = 8$.

9 **a** Find the equations of the tangents to the parabola $y = 2x^2$ at the points where the line $6x - 8y + 1 = 0$ intersects with the parabola.

b Show that the tangents are perpendicular.

10 Find any x values of the function $f(x) = \dfrac{2}{x^3 - 8x^2 + 12x}$ where it is not differentiable.

11 EXT1 A spherical balloon is being inflated so that the surface area is increasing at the rate of $0.3 \text{ cm}^2 \text{ s}^{-1}$. When the balloon's radius is 5 cm, find the rate of increase in:

a the radius

b the volume.

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12 Find the equation of the chord joining the points of contact of the tangents to the curve $y = x^2 - x - 4$ with gradients 3 and –1.

13 For the function $f(x) = ax^2 + bx + c$, $f(2) = 4$, $f'(1) = 0$ and $f'(-3) = 8$. Evaluate a, b and c.

14 For the function $f(x) = x^3$:

a Show that $f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3$.

b Show that $f'(x) = 3x^2$ by differentiating from first principles.

15 Consider the function $f(x) = \frac{1}{x}$.

a Find the gradient of the secant between:

i $f(1)$ and $f(1.1)$ **ii** $f(1)$ and $f(1.01)$ **iii** $f(1)$ and $f(0.99)$

b Estimate the gradient of the tangent to the curve at the point where $x = 1$.

c Show that $\frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$.

d Hence show that $f'(x) = -\frac{1}{x^2}$ by differentiating from first principles.

16 EXT1 A container is being filled with sand such that its volume at any one time is given by $V = 21x + x^2$ cm^3, where x is the depth of sand in the container. If sand is poured into the container at a constant rate of 15 cm^3 per second, at what rate, correct to 2 decimal places, will the level of sand be rising when the depth is 20 cm?

17 The displacement of a particle is given by $x = (t^3 + 1)^6$, where x is in metres and t is in seconds.

a Find the initial displacement and velocity of the particle.

b Find its acceleration after 2 s in scientific notation, correct to 3 significant figures.

c Show that the particle is never at the origin.

18 EXT1 For each function:

i find the inverse function $f^{-1}(x)$ with x in terms of y

ii write $f^{-1}(x)$ with y in terms of x

iii find $\frac{dy}{dx}$ of the inverse function

iv find $\frac{dx}{dy}$ of the inverse function in terms of x

v show $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

a $y = 3x + 1$ **b** $f(x) = 3x^5$